THE CROSSING NUMBERS OF PRODUCTS WITH CYCLES

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ABSTRACT

The crossing numbers of Cartesian products of all graphs of order at most four with cycles are known. The crossing numbers of Cartesian products $G \square C_n$ for several graphs G on five and six vertices and the cycle C_n are also given. In this paper, we extend these results by determining crossing numbers of Cartesian products $G \square C_n$ for some specific six vertex graphs G and for some fixed number $n = 3, 4, 5$.

Keywords: *graph, Cartesian product, crossing number, cycle, drawing*

1. INTRODUCTION

Let G be a simple graph with vertex set V and edge set E. A drawing of the graph in the plane is called a *good drawing* if and only if no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. The *crossing number* $cr(G)$ of a graph G is the minimum number of crossings of edges in a drawing of G in the plane such that no three edges cross in a point. A drawing with minimum number of crossings is always a good drawing.

It is very difficult to establish the crossing number of a given graph. So, the crossing numbers are known only for a few families of graphs. Most of these graphs are Cartesian products of special graphs. The *Cartesian product* $G_1 \square G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \square G_2) = V(G_1) \square V(G_2)$ and edge set $E(G_1 \square G_2) = \{\{(u_i, v_j), (u_k, v_h)\} : (u_i = u_k \text{ and } \{v_j, v_h\} \in E(G_2)) \text{ or } (\{u_i, u_k\} \in E(G_1) \text{ and } v_j = v_h)\}$.

Let C_n be the cycle of length n, P_n be the path of length n, and S_n be the star isomorphic to $K_{1,n}$. Harary et al. [9] conjectured that the crossing number of $C_m \square C_n$ is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. This has been proved only for m, n satisfying $m \leq 7$ [1], [4], [17], [18], [19]. It was recently proved by Glebsky and Salazar [8] that the crossing number of $C_m \square C_n$ equals its long–conjectured value at least for $n \geq m(m+1)$. Beineke and Ringeisen in [2] as well as Jendroľ and Ščerbová in [10] determined the crossing numbers of the Cartesian products of all graphs on four vertices with cycles. Klešč in [11], [12], [13], [14], Klešč, Richter and Stobert in [15], and Klešč and Kocúrová in [16] gave the crossing numbers of $G \square C_n$ for several graphs G of order five.

We are interested in the crossing numbers of Cartesian products of graphs on six vertices with cycles. Except for the star S_5, in [6] there are given the crossing numbers of $G \square C_n$ for all five–edge graphs G on six vertices. In [7],the values of crossing numbers for sevetal Cartesian products of cycles and six–edge graphs G on six vertices are presented. In [7] and [5] are given the crossing numbers for Cartesian products of cycles and two seven–edge graphs G on six vertices. In this paper, we give the crossing number of the Cartesian products $G \square C_n$ for two graphs G on six vertices and fixed number n.

2. THE CROSSING NUMBERS OF $S_5 \square C_3$ AND $S_5 \square C_4$

In [6] there is presented only upper bound $4n$ for the crossing numbers of Cartesian products of star on six vertices with cycles $S_5 \square C_n$ obtained from the drawing of the graph $S_5 \square C_n$ for $n \geq 3$. We suppose that the upper bound in [6] is stated for $n \geq 6$. This bound is lower for $n = 3, 4, 5$. In the next text we determine that $cr(S_5 \square C_3) = 4$ and $cr(S_5 \square C_4) = 8$. The hypothesis about lower bound for $n = 5$, using the drawing of the graph $S_5 \square C_5$, is 16.

Theorem 2.1. $cr(S_5 \square C_3) = 4$*,* $cr(S_5 \square C_4) = 8$*.*

Proof. The graph $S_5 \square C_3$ ($S_5 \square C_4$) contains the graph $S_5 \square P_2$ ($S_5 \square P_3$) as a subgraph. Bokal [3] proved that $cr(S_5 \square P_n) = 4(n-1)$. Thus $cr(S_5 \square C_3) \geq 4$ ($cr(S_5 \square C_4) \geq 8$). In Fig. 1 there are good drawings of $S_5 \square C_3$ and $S_5 \square C_4$ with four and eight crossings, respectively, therefore $cr(S_5 \square C_3) \leq 4$ and $cr(S_5 \square C_4) \leq 8$. □

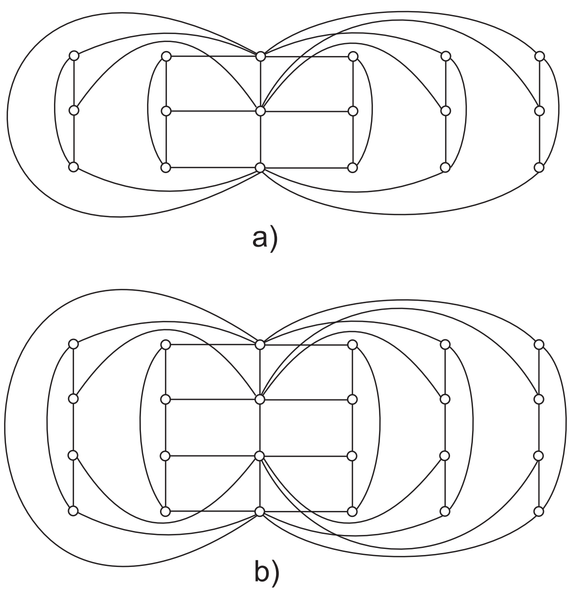

Fig. 1 The drawings of the graphs $S_5 \square C_3$ and $S_5 \square C_4$

3. THE CROSSING NUMBERS OF $G \square C_N$ FOR THE SPECIFIC SIX–EDGE GRAPH G AND FOR $N = 3, 4, 5$

For $n \geq 6$ - what is the upper boud? What about the exact value of the crossing number in this case?

At least - formulate the hypothesis.

In this section, we give the crossing numbers of the Cartesian products $G \square C_3$, $G \square C_4$ and $G \square C_5$ for the graph G shown in Fig. 2. We prove, that $cr(G \square C_3) = 5$, $cr(G \square C_4) = 10$ and $cr(G \square C_5) = 14$. Fig. 3 shows the drawing of the graph $G \square C_n$ in which the edges of every subgraph isomorphic to G are crossed exactly three times. Hence, the crossing number of $G \square C_n$ for $n \geq 6$ is at most $3n$, we conjecture that it is exactly $3n$.

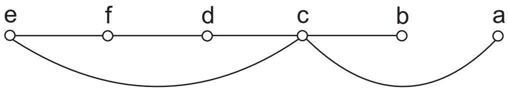

Fig. 2 The graph G

Let D be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge–disjoint subgraphs of G. We denote by $cr_D(G_i, G_j)$ the number of crossings among edges of G_i and edges of G_j, and by $cr_D(G_i)$ the number of crossings between edges of G_i in D.

Assume $n \geq 3$, and consider the graph $G \square C_n$ in the following way: it has $6n$ vertices and edges that are the edges in the n copies G^i, $i = 0, 1, \ldots, n-1$, and in the six cycles of length n. For $i = 0, 1, \ldots, n-1$, let a_i and b_i be the vertices of G^i of degree one, c_i the vertex of degree four and let d_i, e_i and f_i be the vertices of G^i of degree two (see Fig. 3). Thus, for $x \in \{a, b, c, d, e, f\}$, the n–cycle C_n^x is induced by the vertices $x_0, x_1, \ldots, x_{n-1}$.

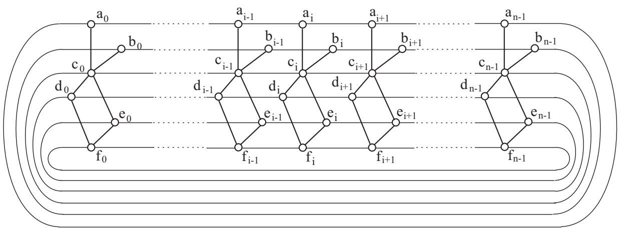

Fig. 3 The drawing of the graph $G \square C_n$

Let T^x, $x \in \{a, b, d, e\}$, be the subgraph of the graph $G \square C_n$ consisting of the cycle C_n^x together with the vertices of C_n^c and of the edges joining C_n^x with C_n^c. Let X^f be the subgraph of $G \square C_n$ induced by the edges incident with the vertices of C_n^f. It is easy to see that T^a, T^b, T^d, T^e, C_n^c, and X^f are edge-disjoint subgraphs and that

$$G \square C_n = T^a \cup T^b \cup C_n^c \cup T^d \cup T^e \cup X^f.$$

The subgraph $T^a \cup T^b \cup C_n^c \cup T^d \cup T^e$ of the graph $G \square C_n$ is isomorphic to the graph $S_4 \square C_n$ and the subgraph $C_n^c \cup T^d \cup T^e \cup X^f$ of the graph $G \square C_n$ is isomorphic to the graph $C_4 \square C_n$.

Theorem 3.1. $cr(G \square C_3) = 5$.

Proof. Fig. 4 shows the good drawing of the graph $G \square C_3$ with five crossings, thus $cr(G \square C_3) \leq 5$.

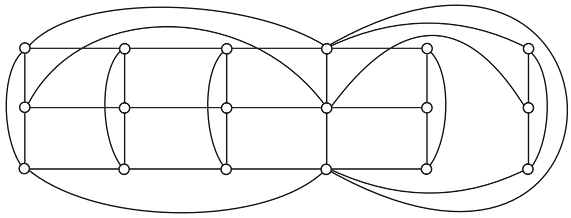

Fig. 4 The drawing of the graph $G \square C_3$

Assume that there is a good drawing of $G \square C_3$ with at most 4 crossings and let D be such a drawing. The subgraph $C_3^c \cup T^d \cup T^e \cup X^f$ of the graph $G \square C_3$ is isomorphic to the graph $C_4 \square C_3$ and $cr(C_4 \square C_3) = 4$ (see [19]). Thus, in D there is no crossing on the edges of $T^a \cup T^b$. The planar subdrawing of $T^a \cup T^b$ induced by D is unique within isomorphism and divides the plane into two triangular and three hexagonal regions in such a way that there is no region with all three vertices c_0, c_1, and c_2 on its boundary. So, an edge of T^d crosses in D an edge of $T^a \cup T^b$, which contradicts the assumption that no edge of $T^a \cup T^b$ is crossed. □

Theorem 3.2. $cr(G \square C_4) = 10$.

Proof. In Fig. 5 there is a good drawing of $G \square C_4$ with ten crossings, thus $cr(G \square C_3) \leq 10$.

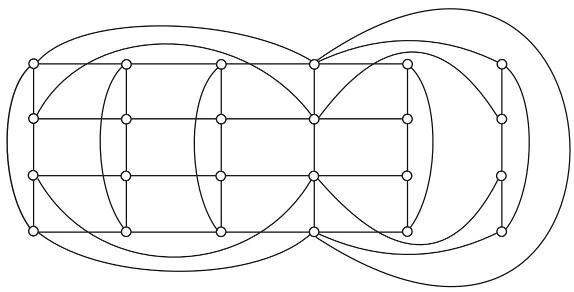

Fig. 5 The drawing of the graph $G \square C_4$

Assume that there is a good drawing of $G \square C_4$ with at most 9 crossings and let D be such a drawing. The graph $G \square C_4$ contains the subgraph $C_4^c \cup T^d \cup T^e \cup X^f$ which is isomorphic to the graph $C_4 \square C_4$ and $cr(C_4 \square C_4) = 8$ (see [4]). Thus, in D there is at most one crossing on the edges of $T^a \cup T^b$. Consider the subgraph $T^a \cup T^b$ of the graph $G \square C_4$ and let D' be its subdrawing induced by D.

First, suppose that $cr_D(T^a \cup T^b) = 0$. As $T^a \cup T^b$ is subdivision of the planar graph $P_1 \square C_4$, the planar subdrawing of $T^a \cup T^b$ induced by D is unique within isomorphism and divides the plane into two quadrangular and four hexagonal regions in such a way that there are at most two of the vertices c_0, c_1, c_2, and c_3 on the boundary of every region.

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So, in D, the edges of T^d cross the edges of $T^a \cup T^b$ at least twice and it contradicts our assumption.

Next, let $cr_D(T^a \cup T^b) = 1$. The subgraph $T^a \cup T^b$ is obtained from $C_4 \square P_1$ by an elementary subdivision of every edge joining two 4–cycles C_4^a and C_4^b and for the graph $C_4 \square P_1$ there is no good drawing with exactly one crossing, because for any two edges which cross each other one can find two vertex-disjoint cycles such that crossed edges are in different cycles. Therefore two vertex-disjoint cycles cannot cross only once, the only one crossing in D' is between an edge incident with a vertex of degree two and an edge of the cycle C_4^a or the cycle C_4^b. In this case, the cycle C_4^a or the cycle C_4^b separates in D some vertex c_i of the cycle C_4^c from the other vertices of C_4^c. Hence, C_4^c crosses in D the edges of $T^a \cup T^b$ at least twice and this contradiction completes the proof. □

Theorem 3.3. $cr(G \square C_5) = 14$.

Proof. In the drawing of the graph $G \square C_5$ in Fig. 6 one can easily see that $cr(G \square C_5) \leq 14$.

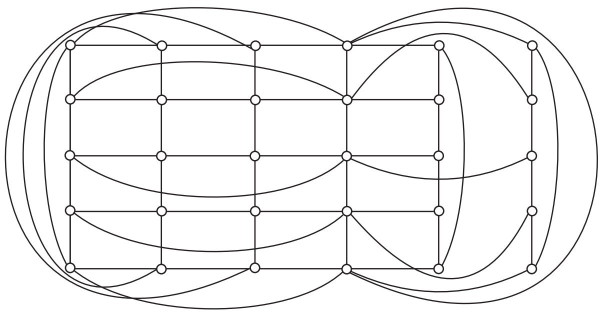

Fig. 6 The drawing of the graph $G \square C_5$

Assume that there is a good drawing of the graph $G \square C_5$ with at most 13 crossings and let D be such a drawing. The graph $G \square C_5$ contains the graph $C_4 \square C_5$ as a subgraph and $cr(C_4 \square C_5) = 10$ (see [2]). Thus, in D there are at most three crossings on the edges of $T^a \cup T^b$. Consider the subgraph $T^a \cup T^b$ of the graph $G \square C_5$ and let D' be its subdrawing induced by D.

First, assume that $cr_D(T^a \cup T^b) = 0$. As $T^a \cup T^b$ is a subdivision of the planar graph $P_1 \square C_5$, the subdrawing D' of $T^a \cup T^b$ induced by D divides the plane into two regions without vertices of C_5^c on their boundaries and into five regions having two vertices of C_5^c on the boundary of every region. If, in D, the cycle C_5^d is placed in a region of D' with fewer than two vertices of C_5^c on its boundary, then $cr_D(T^a \cup T^b, T^d) \geq 5$. If C_5^d is placed in a region with two vertices of C_5^c on the boundary, then one vertex of C_5^c is separated from C_5^d by at least two vertex-disjoint cycles. Hence, $cr_D(T^a \cup T^b, T^d) \geq 4$. If the cycle C_5^d crosses the edges of $T^a \cup T^b$ two or three times, then it is placed in two regions of D' with at most three vertices of C_5^c on their boundaries and, in D, the edges of T^d cross the edges of $T^a \cup T^b$ at least four times. If there are four vertices of C_5^c on the boundaries of the regions in D' in which C_5^d is placed in D, the edges of C_5^d cross the edges of $T^a \cup T^b$ at least four times.

In case 2, assume that $cr_D(T^a \cup T^b) = 1$. As the subgraph $T^a \cup T^b$ is obtained from $P_1 \square C_5$ by elementary subdivision of every edge joining two 5–cycles C_5^a and C_5^b, and therefore for the graph $C_5 \square P_1$ there is no good drawing with exactly one crossing (because for any two edges which cross each other one can find two vertex-disjoint cycles such that crossed edges are in different cycles and two vertex-disjoint cycles cannot cross only once), the only one crossing in D' is between an edge incident with a vertex of degree two and an edge of the cycle C_5^a or the cycle C_5^b. In this case, the cycle C_5^a or the cycle C_5^b separates in D some vertex c_i of the cycle C_5^c from the other vertices of C_5^c. Hence, C_5^c crosses in D the edges of $T^a \cup T^b$ at least twice. The removing of the separated vertex c_i of the cycle C_5^c from D' we have the drawing without crossings. This drawing divides the plane in such a way that there are at most two vertices of C_5^c on the boundary of every region. As the vertex c_i is in D' separated from the other vertices of C_5^c, in the subdrawing D' of $T^a \cup T^b$ with one crossings there are at most two vertices of C_5^c on the boundary of a region. If the cycle C_5^d of T^d crosses the 2-connected subgraph $T^a \cup T^b$, it crosses $T^a \cup T^b$ at least two times. Otherwise C_5^d is in D placed in one region in the view of the subdrawing of $T^a \cup T^b$ and at least two edges of T^d joining C_5^d with the vertices of C_5^c cross the edges of $T^a \cup T^b$. So, in this case, again, there are more than three crossings on the edges of $T^a \cup T^b$. It is a contradiction.

In case 3, assume that $cr_D(T^a \cup T^b) \geq 2$. Then at least one subgraph T^d or T^e does not cross in D the edges of $T^a \cup T^b$. Without loss of generality, let T^d does not cross the edges of $T^a \cup T^b$. So, $cr_D(T^a \cup T^b, T^d) = 0$. In this case, $cr_D(T^a, T^d) = 0$ and $cr_D(T^b, T^d) = 0$. As $T^a \cup T^d$ is a subdivision of the planar graph $P_1 \square C_5$, the subdrawing D'' of $T^a \cup T^d$ divides the plane into several regions without vertices of C_5^c on their boundaries and into regions, which have exactly two vertices of C_5^c on the boundary of one region. Fig. 7 shows the drawing D'' in which possible crossings among the edges of T^a are inside the left disc bounded by the dotted cycle and possible crossings among the edges of T^d are inside the right disc bounded by the dotted cycle.

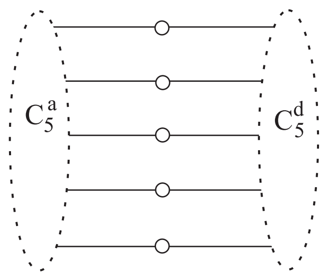

Fig. 7 The subdrawing of the subgraph $T^a \cup T^d$

We can suppose that if, in D, an edge not incident with a vertex of C_5^a or C_5^d passes through one of these two discs, then it crosses the edges of $T^a \cup T^d$ at least twice. Consider now a subgraph T^b. Both C_5^b and $T^a \cup T^d$ are 2–connected graphs and so, $cr_D(C_5^b, T^a \cup T^d) \neq 1$. If, in D, the cycle C_5^b is placed in a region of D'' with fewer than two vertices of C_5^c on its boundary, then $cr_D(T^a \cup T^d, T^b) \geq 4$. If C_5^b is

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placed in a region with two vertices of C_5^c on the boundary, then one vertex of C_5^c is separated from C_5^b by at least two vertex–disjoint cycles. Hence, $cr_D(T^a \cup T^d, T^b) \geq 4$. If the cycle C_5^b crosses the edges of $T^a \cup T^d$ two or three times, then it is placed in two regions of D'' with at most three vertices of C_5^c on their boundaries and the in D edges joining C_5^b with C_5^c cross the edges of $T^a \cup T^b$ at least four times. If there are four vertices of C_5^c on the boundaries of the regions in D'' in which C_5^b is placed in D, at least four crossings between the edges of C_5^b and the edges of $T^a \cup T^d$ are necessary. As $cr_D(T^d, T^b) = 0$, all considered crossings are between the edges of T^a and the edges of T^b. This contradiction with the assumption that there are at most three crossings on the edges of $T^a \cup T^b$ completes the proof. □

4. DISCUSSION/CONCLUSIONS

There are open problems to detemine the crossing numbers of graphs $S_5 \square C_n$ for $n \geq 5$ and of graphs $G \square C_n$ for $n \geq 6$.

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