NUMERICAL-ANALYTIC INVESTIGATION OF SOLUTIONS OF NON-LINEAR INTEGRAL BOUNDARY VALUE PROBLEMS

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ABSTRACT

We consider the integral boundary-value problem for a certain class of non-linear system of ordinary differential equations.

We give a new approach for studying this problem, namely by using an appropriate parametrization technique the given problem is reduced to the equivalent parametrized two-point boundary-value problem with linear boundary conditions without integral term.

To study the transformed problem we use a method based upon a special type of successive approximations, which are constructed analytically.

Keywords: *integral boundary-value problem, successive approximations, parametrization, existence*

1. INTRODUCTION

We show how an appropriate parametrization technique and a suitable successive approximations can help to investigate the solutions of non-linear boundary value problems with integral boundary conditions.

Recently, integral boundary-value problems for non-linear differential equations have attracted a lot of attention [1, 2]. There were studied, mainly, scalar non-linear differential equations of the special form. According our best knowledge, there are only a few works deal with the investigation of the systems of non-linear differential equations of the general form with integral boundary restrictions [3, 4].

The aim of this paper is to extend the numerical-analytic technique, which had been used earlier successfully in relation to different types of boundary-value problems with two- and multi-point linear and non-linear boundary conditions for a class of non-linear differential systems of the form

$$\frac{dx(t)}{dt} = f(t, x(t)),$$

under the integral boundary conditions

$$Ax(0) + \int_0^T B(s)x(s)ds + Cx(T) = d,$$

where A and $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{22} & O_{n-p} \end{pmatrix}$ are some given singular $n \times n$ matrices, where C_{11} is a $p \times p$ matrix, $\det C_{11} \neq 0$, C_{12} is a $p \times (n-p)$ matrix, C_{22} is a $(n-p) \times p$ matrix, O_{n-p} is a $(n-p) \times (n-p)$ zero-matrix and $B(\cdot)$ is a continuous $n \times n$ matrix.

We use a new approach for studying this problem, namely an appropriate parametrization technique. Using it, we can reduce the given problem to the equivalent parametrized two-point boundary-value problem with linear boundary conditions without integral term.

To study the transformed problem, we use a method based upon a special type of successive approximations, which are constructed analytically. We give a sufficient conditions for the uniformly convergence of this sequence and introduce a certain finite-dimensional 'determining' system of algebraic or transcendental equations whose solutions give all the initial values of the solutions of the given boundary-value problem. Based upon the properties of the functions of the constructed sequence and of the determining equations, using the Brower degree, we give efficient conditions ensuring the existence of the original integral boundary-value problem.

We note that the operations $|\cdot|$, $\geq$, $\leq$, max, min between matrixes, vectors and vector functions will be understood componentwise.

2. PROBLEM SETTING

We consider the nonlinear boundary-value problem subjected to the integral boundary conditions

$$\frac{dx(t)}{dt} = f(t, x(t)), \tag{1}$$

$$Ax(0) + \int_0^T B(s)x(s)ds + Cx(T) = d, \tag{2}$$

where A is arbitrary and C is some given singular $n \times n$ matrix of the form $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{22} & O_{n-p} \end{pmatrix}$, where C_{11} is a $p \times p$ matrix, $\det C_{11} \neq 0$, C_{12} is a $p \times (n-p)$ matrix, C_{22} is a $(n-p) \times p$ matrix, O_{n-p} is a $(n-p) \times (n-p)$ zero-matrix and $B(\cdot)$ is a continuous $n \times n$ matrix.

Here, we suppose that the vector function

$$f : [0, T] \times D \to \mathbb{R}^n$$

is continuous, where $D \subset \mathbb{R}^n$ is a closed and bounded domain, and let us put

$$D_0 := \left\{ \int_0^T B(s)x(s)ds : x \in C([0, T]; D) \right\}.$$

The problem is to find the solution of the system of differential equations (1) satisfying integral boundary restrictions (2) in a class of continuously-differentiable vector functions $x : [0, T] \to D$.

3. PARAMETRIZATION OF THE INTEGRAL BOUNDARY CONDITIONS

To pass to the linear two-point boundary conditions from (2), similarly to [5–8] we introduce the following parameters

$$\begin{aligned} z &:= x(0) = \operatorname{col}(x_1(0), x_2(0), \dots, x_n(0)) = \\ &= \operatorname{col}(z_1, z_2, \dots, z_n), \\ \lambda &:= \int_0^T B(s)x(s)ds = \operatorname{col}(\lambda_1, \lambda_2, \dots, \lambda_n), \end{aligned}$$

$$\eta := \operatorname{col}\left(\underbrace{0,0,\dots,0,}_{p} x_{p+1}(T), x_{p+2}(T), \dots, x_n(T)\right) = \\ = \operatorname{col}\left(\underbrace{0,0,\dots,0,}_{p} \eta_{p+1}, \eta_{p+2}, \dots, \eta_n\right). \quad (3)$$

Using parametrization (3), the integral boundary restrictions (2) can be written as the linear ones:

$$Ax(0) + C_1x(T) = d - \lambda + \eta, \quad (4)$$

where $C_1 = \begin{pmatrix} C_{11} & C_{12} \\ C_{22} & I_{n-p} \end{pmatrix}$, I_{n-p} is a $(n-p)\times(n-p)$ unit matrix, λ and η are parameters given by (3).

Let us put:

$$d(\lambda,\eta) := d - \lambda + \eta. \quad (5)$$

Taking into account (5) the parametrized boundary conditions (4) can be rewritten in the form:

$$Ax(0) + C_1x(T) = d(\lambda,\eta). \quad (6)$$

So, instead of the original boundary-value problem with integral boundary conditions (1), (2) we study an equivalent parametrized one, containing already linear boundary restrictions (1), (6). It should be noted that the matrix C_1 in (6) is already non-singular.

Remark 3.1. *The set of the solutions of the non-linear boundary-value problem with integral boundary conditions (1), (2) coincides with the set of the solutions of the parametrized problem (1) with linear boundary restrictions (6), satisfying additional conditions (3).*

4. CONSTRUCTION OF THE SUCCESSIVE APPROXIMATIONS

Let us introduce the vector

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\right], \quad (7)$$

and suppose that the original boundary-value problem (1), (2) is such that the subset

$$D_\beta := \left\{z \in D : B\left(z + \frac{t}{T}C_1^{-1}[d(\lambda,\eta) - (A+C_1)z], \frac{T}{2}\delta_D(f)\right)\right\}$$

$\forall\lambda \in D_0$, $\eta \in D$ is non-empty

$$D_\beta \neq \emptyset. \quad (8)$$

It means that the collection of the points

$$z + \frac{t}{T}C_1^{-1}[d(\lambda,\eta) - (A+C_1)z]$$

belongs to the domain D together with their

$$\beta = \frac{T}{2}\delta_D(f)$$

neighbourhood, $\forall\lambda \in D_0$, $\eta \in D$.

Assume that the function $f(t,x)$ in the right hand-side of (1) satisfies Lipschitz condition of the form

$$|f(t,u) - f(t,v)| \le K|u-v|, \quad (9)$$

for all $t \in [0,T]$, $\{u,v\} \subset D$ with some non-negative constant matrix $K = (k_{ij})_{i,j=1}^n$.

Moreover, we suppose that the spectral radius $r(K)$ of the matrix K satisfies the following inequality

$$r(K) < \frac{10}{3T}. \quad (10)$$

Let us connect with the parametrized boundary-value problem (1), (6) the sequence of functions:

$$x_m(t,z,\lambda,\eta) := z + \int_0^t f(s, x_{m-1}(s,z,\lambda,\eta))ds - \\ - \frac{t}{T}\left[\int_0^T f(s, x_{m-1}(s,z,\lambda,\eta))ds + \right. \\ \left. + C_1^{-1}[d(\lambda,\eta) - (A+C_1)z]\right], \quad (11)$$

where $m = 1,2,3,\dots$,

$$x_0(t,z,\lambda,\eta) = z + \frac{t}{T}C_1^{-1}[d(\lambda,\eta) - (A+C_1)z] \in D_\beta,$$

and z, λ, η are considered as parameters.

It is easy to check that the functions $x_m(t,z,\lambda,\eta)$ satisfy linear parametrized boundary conditions (6) for all $m \ge 1$, z, η, $\lambda \in \mathbb{R}^n$.

The following statement establishes the convergence of the sequence (11).

Theorem 4.1. *Assume that the function $f : [0,T]\times D \to \mathbb{R}^n$ in the right hand-side of the system of differential equations (1) and the parametrized boundary restrictions (6) satisfy conditions (8)–(10).*

Then for all fixed $z \in D_\beta$, $\lambda \in D_0$, $\eta \in D$:

1. *The functions of the sequence (11) are continuously differentiable and satisfy the parametrized boundary conditions (6):*

   $$Ax_m(0,z,\lambda,\eta) + C_1x_m(T,z,\lambda,\eta) = d(\lambda,\eta),$$

   m=1,2,3,....

2. *The sequence of functions (11) for $t \in [0,T]$ converges uniformly as $m \to \infty$ to the limit function*

   $$x^*(t,z,\lambda,\eta) = \lim_{m\to\infty} x_m(t,z,\lambda,\eta). \quad (12)$$

3. *The limit function $x^*(t,z,\lambda,\eta)$ satisfies the parametrized linear two-point boundary conditions:*

   $$Ax^*(0,z,\lambda,\eta) + C_1x^*(T,z,\lambda,\eta) = d(\lambda,\eta).$$

4. *The limit function (12) for all $t \in [0,T]$ is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_0^t f(s,x(s))ds - \frac{t}{T}\left[\int_0^T f(s,x(s))ds + \right.$$
$$\left. + C_1^{-1}\left[d(\lambda,\eta) - (A+C_1)z\right]\right], \quad (13)$$

i. e., it is the solution of the Cauchy problem for the modified system of differential equations:

$$\frac{dx}{dt} = f(t,x) + \Delta(z,\lambda,\eta), \quad (14)$$

$$x(0) = z, \quad (15)$$

where

$$\Delta(z,\lambda,\eta) := \frac{1}{T}\left[C_1^{-1}\left[d(\lambda,\eta) - \right.\right.$$
$$\left.\left. - (A+C_1)z\right] - \int_0^T f(s,x(s))ds\right]. \quad (16)$$

5. *The following error estimation holds:*

$$|x^*(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)| \leq$$
$$\leq \frac{20}{9}t\left(1-\frac{t}{T}\right)Q^m(I_n - Q)^{-1}\delta_D(f), \quad (17)$$

where matrix

$$Q := \frac{3T}{10}K. \quad (18)$$

Consider the Cauchy problem

$$\frac{dx}{dt} = f(t,x) + \mu, \quad t \in [0,T], \quad (19)$$

$$x(0) = z, \quad (20)$$

where $\mu = \mathrm{col}(\mu_1,\ldots,\mu_n)$ is a control parameter.

Theorem 4.2. *Let $z \in D_\beta$, $\lambda \in D_0$, $\eta \in D$ and $\mu \in \mathbb{R}^n$ — are some given vectors. Suppose that for the system of differential equations (1) all conditions of Theorem 4.1 are hold.*

Then in order the solution $x = x(t,z,\lambda,\eta,\mu)$ of the initial-value problem (19), (20) also satisfies parametrized boundary conditions (6) it is necessary and sufficient that the parameter μ was given by formula

$$\mu = \mu_{z,\lambda,\eta} = \frac{1}{T}\left[C_1^{-1}[d(\lambda,\eta) - (A+C_1)z] - \right.$$
$$\left. - \int_0^T f(s,x^*(s,z,\lambda,\eta))\right]ds. \quad (21)$$

In this case

$$x(t,z,\lambda,\eta,\mu) = x^*(t,z,\lambda,\eta) = \lim_{m\to\infty} x_m(t,z,\lambda,\eta), \quad (22)$$

where $x_m(\cdot,z,\lambda,\eta)$ is a sequence of functions defined by (11).

Let's find out the relation of the limit function $x = x^*(t,z,\lambda,\eta)$ of the sequence (11) to the solution of the parametrized two-point boundary-value problem (1) with linear boundary conditions (6) or the equivalent non-linear problem (1) with integral conditions (2).

Theorem 4.3. *Let the conditions (8)–(10) are hold for the original boundary-value problem (1), (2).*

Then the limit function $x^*(\cdot,z^*,\lambda^*,\eta^*)$ is the solution of the parametrized boundary-value problem (1), (6) if and only if

$$z^* = (z_1^*, z_2^*, \ldots, z_n^*),$$
$$\eta^* = (\underbrace{0,0,\ldots,0}_{p}, \eta_{p+1}^*, \eta_{p+2}^*, \ldots, \eta_{p+n}^*),$$
$$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$$

satisfy determining system of algebraic or transcendental equations

$$\Delta(z,\lambda,\eta) = \frac{1}{T}\left[C_1^{-1}[d(\lambda,\eta) - (A+C_1)z] - \right.$$
$$\left. - \int_0^T f(s,x^*(s,z,\lambda,\eta))ds\right] = 0, \quad (23)$$

$$\int_0^T B(s)x^*(s,z,\lambda,\eta)ds = \lambda, \quad (24)$$

$$x_i^*(T,z,\lambda,\eta) = \eta_i, \quad (25)$$
$$i = \overline{p+1,n}.$$

The next statement proves that the system of determining equations (23)–(25) defines all possible solutions of the original non-linear boundary-value problem (1) with integral boundary restrictions (2).

Lemma 4.1. *Let all conditions of Theorem 4.1 be satisfied. Furthermore there exist some vectors $z \in D_\beta$, $\lambda \in D_0$ and $\eta \in D$ that satisfy the system of determining equations (23)–(25). Then the non-linear boundary-value problem (1) with integral boundary conditions (2) has the solution $x(\cdot)$ such that*

$$x(0) = z,$$
$$\textstyle\int_0^T B(s)x(s)ds = \lambda,$$
$$x_i(T) = \eta_i,$$
$$i = \overline{p+1,n}.$$

Moreover this solution is given by formula

$$x(t) = x^*(t,z,\lambda,\eta), \quad t = [0,T], \quad (26)$$

where $x^*(t,z,\lambda,\eta)$ is the limit function of the sequence (11).

And if the boundary-value problem (1), (2) has a solution $x(\cdot)$, then this solution is given by (26), and the system of determining equations (23)–(25) is satisfied when

$$z = x(0),$$
$$\lambda = \textstyle\int_0^T B(s)x(s)ds,$$
$$\eta_i = x_i(T),$$
$$i = \overline{p+1,n}.$$

Remark 4.1. *The main difficulty of realization of this method is to find the limit function $x^*(\cdot,z,\lambda,\eta)$. But in most cases this problem can be solved using the properties of the approximate solution $x_m(\cdot,z,\lambda,\eta)$ built in an analytic form.*

For $m \geq 1$ let us define a function

$$\Delta_m : D_\beta \times D_0 \times D \to \mathbb{R}^n$$

by formula

$$\Delta_m(z,\lambda,\eta) := \frac{1}{T}\left[C_1^{-1}\left[d(\lambda,\eta)-(A+C_1)z\right]- \right.$$
$$\left. -\int_0^T f(s,x_m(s,z,\lambda,\eta))ds\right],\ m=1,2,3,\ldots, \quad (27)$$

where z, λ and η are given by the relation (3).

To investigate the solvability of the parametrized boundary-value problem (1), (6) we observe an approximate determining system of algebraic or transcendental equations of the form

$$\Delta_m(z,\lambda,\eta) = \frac{1}{T}\left[C_1^{-1}\left[d(\lambda,\eta)-(A+C_1)z\right]- \right.$$
$$\left. -\int_0^T f(s,x_m(s,z,\lambda,\eta))ds\right] = 0, \quad (28)$$

$$\int_0^T B(s)x_m(s,z,\lambda,\eta)\,ds = \lambda, \quad (29)$$

$$x_{m,i}(T,z,\lambda,\eta) = \eta_i, \quad (30)$$

$i=\overline{p+1,n}$, where $x_m(\cdot,z,\lambda,\eta)$ is a vector-function, that defines with the recursive relation (11). Increasing m systems (23)–(25) and (28)–(30) are close enough to provide needed precision of finding an approximate solution of the original boundary-value problem (1), (2).

5. EXISTENCE OF THE SOLUTIONS OF THE INTEGRAL BOUNDARY-VALUE PROBLEM

Lemma 5.1. *Let conditions of Theorem 4.1 be satisfied.*

Then for arbitrary $m \geq 1$ and z, λ, η of the form (3) for exact and approximate determining functions

$$\Delta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$\Delta_m : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

from (16) and (27), the estimate

$$|\Delta(z,\lambda)-\Delta_m(z,\lambda)| \leq \frac{10T}{27}KQ^m(I_n-Q)^{-1}\delta_D(f), \quad (31)$$

is true, where K, Q, $\delta_D(f)$ are given correspondingly by (9), (18) (7).

Lemma 5.2. *Let conditions of Theorem 4.1 be satisfied.*

Then for arbitrary $m \geq 1$ and z, λ, η of the form (3) for the functions $x^*(t,z,\lambda,\eta)$ and $x_m(t,z,\lambda,\eta)$ correspondingly of the form (12) and (11) the following estimate

$$\left|\int_0^T B(s)x^*(s,z,\lambda,\eta)\,ds - \int_0^T B(s)x_m(s,z,\lambda,\eta)\,ds\right| \leq$$
$$\leq \frac{10}{9}\bar{B}Q^m(I_n-Q)^{-1}\delta_D(f) \quad (32)$$

is true, where Q, $\delta_D(f)$ are given correspondingly by (18), (7) and

$$\bar{B} = \int_0^T |B(s)|\alpha_1(s)ds.$$

On the base of equations (23)–(25) and (28)–(30) let us introduce the mappings:

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{3n},$$
$$\Phi_m : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{3n},$$

by setting for all z, λ, η of form (3)

$$\Phi(z,\lambda,\eta) := \begin{pmatrix} \frac{1}{T}\left[C_1^{-1}\left[d(\lambda,\eta)-(A+C_1)z\right]- \right. \\ \left. -\int_0^T f(s,x^*(s,z,\lambda,\eta))\,ds\right] \\ \int_0^T B(s)x^*(s,z,\lambda,\eta)\,ds-\lambda \\ x_i^*(T,z,\lambda,\eta)-\eta_i \end{pmatrix}, \quad (33)$$

$$\Phi_m(z,\lambda,\eta) := \begin{pmatrix} \frac{1}{T}\left[C_1^{-1}\left[d(\lambda,\eta)-(A+C_1)z\right]- \right. \\ \left. -\int_0^T f(s,x_m(s,z,\lambda,\eta))\,ds\right] \\ \int_0^T B(s)x_m(s,z,\lambda,\eta)\,ds-\lambda \\ x_{m,i}(T,z,\lambda,\eta)-\eta_i \end{pmatrix}, \quad (34)$$

$i=\overline{p+1,n}$.

Definition 5.1. *Let $H \subset \mathbb{R}^{3n}$ be an arbitrary non-empty set. For any pair of functions*

$$f_j = \operatorname{col}\left(f_{j1}(x),\ldots,f_{j,3n}(x)\right) : H \to \mathbb{R}^{3n}, \quad j=1,2$$

we write

$$f_1 \rhd_H f_2$$

if and only if there exist a function

$$k : H \to \{1,2,\ldots,3n\}$$

such that

$$f_{1,k(x)} > f_{2,k(x)}$$

for all $x \in H$, which means that at every point $x \in H$ at least one of the components of the vector $f_1(x)$ is greater then the corresponding component of the vector $f_2(x)$.

Let us consider the set

$$\Omega = D_1 \times \Lambda_1 \times D_2, \quad (35)$$

where $D_1 \subset D_\beta$, $\Lambda_1 \subset D_0$, $D_2 \subset D$ — are certain bounded open sets.

Theorem 5.1. *Assume that conditions of Theorem 4.1 hold and, moreover, one can specify an $m \geq 1$ and a set $\Omega \subset \mathbb{R}^{3n}$ of the form (35) such that*

$$|\Phi_m| \triangleright_{\partial\Omega} \begin{pmatrix} \frac{10T}{27} K Q^m (I_n - Q)^{-1} \delta_D(f) \\ \frac{10}{9} \bar{B} Q^m (I_n - Q)^{-1} \delta_D(f) \\ \frac{5T}{9} Q^m (I_n - Q)^{-1} \delta_D(f) \end{pmatrix}, \qquad (36)$$

holds, where $\partial\Omega$ is a bound of domain Ω. If, in addition, the Brower degree of Φ_m over Ω with respect to zero satisfies the inequality

$$\deg(\Phi_m, \Omega, 0) \neq 0, \qquad (37)$$

then there exists a triplex $(z^*, \lambda^*, \eta^*) \in \Omega$ such that the function

$$x^*(t) = x^*(t, z^*, \lambda^*, \eta^*) = \lim_{m\to\infty} x_m(t, z^*, \lambda^*, \eta^*) \qquad (38)$$

is a solution of the nonlinear differential system (1) subjected to the integral boundary conditions (2) with the initial condition

$$x^*(0) = z^*. \qquad (39)$$

Remark 5.1. *The proves of all statements above can be done using the technique from* [9–11].

6. APPLICATION

Consider the system

$$\begin{cases} \dfrac{dx_1}{dt} = 0.05x_2 + x_1x_2 - 0.005t^2 - 0.01t^3 + 0.1, \\ \dfrac{dx_2}{dt} = 0.5x_1 - x_2^2 + 0.01t^4 + 0.15t, \end{cases} \qquad (40)$$

where $t \in \left[0, \frac{1}{2}\right]$, with non-linear two-point integral boundary conditions

$$Ax(0) + \int_0^{\frac{1}{2}} B(s)x(s)ds + Cx\left(\frac{1}{2}\right) = d, \qquad (41)$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & t/2 \\ 1/2 & 1/4 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 13/256 \\ 7/960 \end{pmatrix}.$$

It is easy to check that the exact solution of the problem (40), (41) is

$$\begin{cases} x_1^* = 0.1t, \\ x_2^* = 0.1t^2. \end{cases}$$

Suppose that the boundary-value problem (40), (41) is considered in the domain

$$D = \{(x_1, x_2) : |x_1| \leq 0.42, |x_2| \leq 0.4\}.$$

Let us introduce the following parameters:

$$\begin{aligned} &z := x(0) = \mathrm{col}(x_1(0), x_2(0)) = \mathrm{col}(z_1, z_2), \\ &\lambda := \textstyle\int_0^T B(s)x(s)ds = \mathrm{col}(\lambda_1, \lambda_2), \\ &\eta_2 := x_2\left(\tfrac{1}{2}\right) \end{aligned} \qquad (42)$$

Using (42), the boundary restrictions (41) can be rewritten as linear ones that contain already non-singular matrix C_1

$$Ax(0) + C_1 x\left(\frac{1}{2}\right) = d(\lambda, \eta), \qquad (43)$$

where

$$\eta = \mathrm{col}(0, \eta_2), \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d(\lambda, \eta) := d - \lambda + \eta.$$

It is easy to check that the matrix K from the Lipschitz condition (9) can be taken as

$$K = \begin{pmatrix} 0 & 0.05 \\ 0.5 & 0.8 \end{pmatrix},$$

and

$$r(K) < 0.84 < \frac{10}{3T},$$

when $T = 1/2$.

Vector $\delta_D(f)$ can be chosen as

$$\delta_D(f) \leq \begin{pmatrix} 0.18925 \\ 0.3278125 \end{pmatrix}.$$

Domain D_β is defined by inequalities:

$$2t(0.05078125000 - \lambda_1 - z_1) \leq 0.0473125,$$

$$2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2) \leq 0.081953125,$$

$\forall \lambda_1, \lambda_2 \in D_0$, $\eta_2 \in D$.

The domain D_0 is such that

$$D_0 = \{(\lambda_1, \lambda_2) : |\lambda_1| \leq 0.105, |\lambda_2| \leq 0.31\}.$$

One can verify that, for the parametrized boundary-value problem (40), (43), all needed conditions are fulfilled. So, we can proceed with application of the numerical-analytic scheme described above and thus construct the sequence of approximate solutions.

The computation shows that the approximate solutions of the determining system (28)–(30) for $m = 1$ are

$$\begin{aligned} &z_1 := z_{11} = -4.253290711 \cdot 10^{-7}, \\ &z_2 := z_{12} = 7.295492706 \cdot 10^{-7}, \\ &\lambda_1 := \lambda_{11} = 0.0007814848293, \\ &\lambda_2 := \lambda_{12} = 0.007290937121, \\ &\eta_2 := \eta_{12} = 0.0249993271. \end{aligned}$$

The first approximation to the first and second components of solution is

$$x_{11} = -0.0025t^4 + 0.09968792498t - 4.253290711 \cdot 10^{-7} + \\ + 0.001249955722t^2 - 8.714713042 \cdot 10^{-8}t^3,$$

$$x_{12} = 0.00008047566353t + 0.002t^5 + 7.295492706 \cdot 10^{-7} + \\ + 0.1000000588t^2 - 0.0008332398387t^3.$$

The error of the first approximation is

$$\max_{t\in[0,\frac{1}{2}]} |x_1^*(t) - x_{11}(t)| \le 2.1 \cdot 10^{-5},$$
$$\max_{t\in[0,\frac{1}{2}]} |x_2^*(t) - x_{12}(t)| \le 2.2 \cdot 10^{-6}.$$

The error of the second approximation is

$$\max_{t\in[0,\frac{1}{2}]} |x_1^*(t) - x_{21}(t)| \le 4.03 \cdot 10^{-8},$$
$$\max_{t\in[0,\frac{1}{2}]} |x_2^*(t) - x_{22}(t)| \le 1.2 \cdot 10^{-6}.$$

Continuing calculations one can get more approximate solutions of the original boundary-value problem with higher precision.

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