ROBUSTNESS OF INTERVAL TOEPLITZ MATRICES IN FUZZY ALGEBRA

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ABSTRACT

Fuzzy algebra is an algebraic structure in which classical addition and multiplication are replaced by $\oplus$ *and* $\otimes$*, where* $a \oplus b = \max\{a,b\}$*,* $a \otimes b = \min\{a,b\}$*. Fuzzy discrete dynamic systems can be introduced by fuzzy matrices and are useful for describing knowledge engineering, scheduling, cluster analysis, fuzzy logic programs, diagnosis of technical devices or medical diagnosis.*

The paper deals with robust matrices over fuzzy algebra. There are defined the terms of the possible and universal robustness of interval matrices. The work describes the necessary and sufficient condition for the possible and universal robustness of interval Toeplitz matrices which satisfy a certain condition, called the condition $\mathscr{C}^*$*.*

Keywords: *fuzzy algebra, Toeplitz matrix, interval Toeplitz matrix, possible robustness, universal robustness*

1. INTRODUCTION

Studying matrix properties in fuzzy algebra, where addition and multiplication are formally replaced by operations of maximum and minimum, is of great importance for applications in various areas. Fuzzy discrete dynamic systems can be introduced by fuzzy matrices and are useful for describing knowledge engineering, scheduling, cluster analysis, fuzzy logic programs [6], diagnosis of technical devices [13], [14] or medical diagnosis [12].

Periodic behaviour of fuzzy matrices with corresponding polynomial algorithms were studied in [5] and [10]. However, in practice we deal often with inexact input data. This leads to demand replace scalar matrices by so-called interval matrices ([1]).

The main aim of this paper is to describe so called robust matrices and introduce the necessary and sufficient conditions for the possible and universal robustness of an interval Toeplitz matrices.

2. BACKGROUND OF THE PROBLEM

The fuzzy algebra $\mathscr{B}$ is the triple $(B,\oplus,\otimes)$, where $(B,\leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$ and $\otimes$, respectively. The least element in B will be denoted by O, the greatest one by I.

By $\mathbb{N}$ we denote the set of all natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The greatest common divisor of a set $S \subseteq \mathbb{N}$ is denoted by $\gcd S$ and the least common multiple by $\operatorname{lcm} S$. For a given natural number $n \in \mathbb{N}$, we use the notations $N = \{1,2,\dots,n\}$.

For any $n \in \mathbb{N}$, $B(n,n)$ denotes the set of all square matrices of order n and $B(n)$ denotes the set of all n-dimensional column vectors over $\mathscr{B}$. The matrix operations over $\mathscr{B}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as matrix operations over any field. The r-th power of a matrix A is denoted by A^r, with elements $(A^r)_{ij}$.

For $A = (a_{ij}) \in B(n,n), C = (c_{ij}) \in B(n,n)$ we write $A \leq C$ $(A < C)$ if $a_{ij} \leq c_{ij}$ $(a_{ij} < c_{ij})$ holds for all $i, j \in N$.

By digraph we understand a pair $\mathscr{G} = (V_{\mathscr{G}}, E_{\mathscr{G}})$, where $V_{\mathscr{G}}$ is a non-empty finite set, called the node set, and $E_{\mathscr{G}} \subseteq V_{\mathscr{G}} \times V_{\mathscr{G}}$, called the arc set. A digraph $\mathscr{G}' = (V_{\mathscr{G}'}, E_{\mathscr{G}'})$ is a subdigraph of digraph $\mathscr{G}$, if $V_{\mathscr{G}'} \subseteq V_{\mathscr{G}}$ and $E_{\mathscr{G}'} \subseteq E_{\mathscr{G}}$. Specially, $\mathscr{G}/V_{\mathscr{G}'}$ stands for the subdigraph induced by vertex set $V_{\mathscr{G}'}$.

A walk in a digraph $\mathscr{G}$ is the sequence of nodes and arcs $\mathscr{P} = (v_0, e_1, v_1, e_2, v_2, \dots, v_{l-1}, e_l, v_l)$ such that $e_k = (v_{k-1}, v_k) \in E_{\mathscr{G}}$ for $k = 1,2,\dots,l$. A walk in $\mathscr{G}$ is a trail if all its arcs are distinct. The number l is the length of the trail $\mathscr{P}$ and is denoted by $\ell(\mathscr{P})$. If $v_0 = v_l$, then $\mathscr{P}$ is called a cycle. A cycle is elementary if all nodes except the terminal node are distinct. A digraph is called strongly connected if any two distinct nodes of $\mathscr{G}$ are contained in a common cycle. By a strongly connected component of $\mathscr{G}$ we mean a maximal strongly connected subdigraph of $\mathscr{G}$. A strongly connected component $\mathscr{K} = (V_{\mathscr{K}}, E_{\mathscr{K}})$ is called non-trivial if there is a cycle of positive length in $\mathscr{K}$. For any non-trivial strongly connected component $\mathscr{K}$ is the *period* of $\mathscr{K}$ defined as

$$\operatorname{per}\mathscr{K} = \gcd\{\,\ell(c);\ c \text{ is a cycle in } \mathscr{K},\ \ell(c) > 0\}. \tag{1}$$

If $\mathscr{K}$ is trivial, then $\operatorname{per}\mathscr{K} = 1$. By $\mathrm{SCC}^\star\mathscr{G}$ we denote the set of all non-trivial strongly connected components of $\mathscr{G}$.

Lemma 2.1. *Let* $\mathscr{K} \in \mathrm{SCC}^\star\mathscr{G}$*,* $\mathscr{K}' \in \mathrm{SCC}^\star\mathscr{G}'$ *and* $\mathscr{K} \subseteq \mathscr{K}'$*. Then* $\operatorname{per}\mathscr{K}' \mid \operatorname{per}\mathscr{K}$*.*

Proof. Denote by $C_{\mathscr{K}}$ and $C_{\mathscr{K}'}$ the sets of all cycles in $\mathscr{K}$ and $\mathscr{K}'$, respectively. Since

$$C_{\mathscr{K}} = \{\ell(c);\ c \text{ is a cycle in } \mathscr{K},\ \ell(c) > 0\} \subseteq$$

$$\subseteq \{\ell(c');\ c' \text{ is a cycle in } \mathscr{K}',\ \ell(c') > 0\} = C_{\mathscr{K}'},$$

we obtain

$$\operatorname{per}\mathscr{K}' = \gcd C_{\mathscr{K}'} \mid \gcd C_{\mathscr{K}} = \operatorname{per}\mathscr{K}. \qquad \square$$

Further, we define the *period* of the digraph $\mathscr{G}$ as follows

$$\operatorname{per}\mathscr{G} = \operatorname{lcm}\{\operatorname{per}\mathscr{K};\ \mathscr{K} \in \mathrm{SCC}^\star\mathscr{G}\}.$$

For a given matrix $A \in B(n,n)$ the symbol $\mathscr{G}(A) = (V_{\mathscr{G}(A)}, E_{\mathscr{G}(A)})$ stands for the complete, edge-weighted digraph associated with A, i.e., the vertex set of $\mathscr{G}(A)$ is N, and the capacity of any edge $(i,j) \in E_{\mathscr{G}(A)}$ is a_{ij}. In addition, for given $h \in B$, the *threshold digraph* $\mathscr{G}(A,h)$ is the

digraph with the vertex set $V_{\mathscr{G}(A,h)} = N$ and the edge set $E_{\mathscr{G}(A,h)} = \{(i,j);\ i,j \in N,\ a_{ij} \geq h\}$.
The following lemma describes the relation between matrices and corresponding threshold digraphs and follows from the transitivity of ordering.

Lemma 2.2. [7] *Let $A, C \in B(n,n)$. Let $h, h_1, h_2 \in B$.*

(i) *If $A \leq C$, then $\mathscr{G}(A,h) \subseteq \mathscr{G}(C,h)$,*

(ii) *if $h_1 \leq h_2$, then $\mathscr{G}(A,h_2) \subseteq \mathscr{G}(A,h_1)$.*

Let $A \in B(n,n)$ and $x \in B(n)$. The orbit $\mathscr{O}(A,x)$ of $x = x^{(0)}$ generated by A is the sequence

$$x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots,$$

where $x^{(r)} = A^r \otimes x^{(0)}$ for each $r \in \mathbb{N}$.

For a given matrix $A \in B(n,n)$, the element $\lambda \in B$ and the n-tuple $x \in B(n)$ are the so-called *eigenvalue* of A and *eigenvector* of A, respectively, if

$$A \otimes x = \lambda \otimes x.$$

The *eigenspace* $V(A,\lambda)$ is defined as the set of all eigenvectors of A with associated eigenvalue λ, i.e.,

$$V(A,\lambda) = \{x \in B(n);\ A \otimes x = \lambda \otimes x\}.$$

Let $\lambda \in B$. A matrix $A \in B(n,n)$ is *ultimately λ-periodic* if there are natural numbers p and R such that the following holds:

$$A^{k+p} = \lambda \otimes A^k \quad \text{for all} \quad k \geq R.$$

The smallest natural number p with the above property is called the period of A, denoted by $\operatorname{per}(A,\lambda)$. In case $\lambda = I$ let us denote $\operatorname{per}(A,I)$ by abbreviation $\operatorname{per} A$. It is known that if a matrix is ultimately λ-periodic, then it is ultimately I-periodic and $\operatorname{per}(A,\lambda) = \operatorname{per}(A,I)$.

According to [5] we define

$$\mathrm{SCC}^{\star}(A) = \cup\{\mathrm{SCC}^{\star}\mathscr{G}(A,h);\, h \in \{a_{ij}; i,j \in N\}\}$$

Theorem 2.1. [5] *Let $A \in B(n,n)$. Then*

$$\operatorname{per} A = \operatorname{lcm}\{\operatorname{per}\mathscr{K};\ \mathscr{K} \in \mathrm{SCC}^{\star}(A)\}.$$

Let us denote

$$T(A,\lambda) = \{x \in B(n);\ \mathscr{O}(A,x) \cap V(A,\lambda) \neq \emptyset\}.$$

Definition 2.1. *Let $A = (a_{ij}) \in B(n,n)$, $\lambda \in B$. A matrix A is called λ-robust if $T(A,\lambda) = B(n)$.*

A λ-robust matrix with $\lambda = I$ is called *robust matrix*. It is easy to see that if $A = (a_{ij})$ is ultimately λ-periodic and $\lambda \geq \max\limits_{i,j \in N} a_{ij}$, then $\lambda \geq \max\limits_{i,j \in N} a_{ij}^k$ and $A^{k+p} = \lambda \otimes A^k = A^k$ for $k \geq R$. Hence the necessary condition for $A = (a_{ij})$ to be ultimately λ-periodic is $\lambda \geq \max\limits_{i,j \in N} a_{ij}^k$. The results we shall formulate for $\lambda = I$, as well as the methods used to prove them, can be generalized for arbitrary $\lambda \in [\max\limits_{i,j \in N} a_{ij}, I]$.

We recall a result of the paper [10] adapted for $\lambda = I$.

Lemma 2.3. [10] *Let $A = (a_{ij}) \in B(n,n)$. Then A is robust if and only if* $\operatorname{per} A = 1$.

Note that an $O(n^3)$ algorithm for finding $\operatorname{per} A$ is presented in [5].

3. PERIODICITY OF TOEPLITZ MATRICES

In this section we shall deal with the special class of matrices, the Toeplitz matrices. A Toeplitz matrix contains the same element on every diagonal, which is paralell to the main diagonal.

Definition 3.1. *Let $a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1} \in B$. A matrix $A \in B(n,n)$ of the form*

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & \ldots & a_{n-3} & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \ldots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{-n+2} & a_{-n+3} & a_{-n+4} & \ldots & a_0 & a_1 \\ a_{-n+1} & a_{-n+2} & a_{-n+3} & \ldots & a_{-1} & a_0 \end{pmatrix}$$

is called a Toeplitz matrix. We denote a Toeplitz matrix by

$$A = Tp(a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1}).$$

We denote

(i) $N^- = \{-n+1, -n+2, \ldots, -1\}$,
$N^+ = \{1, 2, \ldots, n-1\}$,
$N^* = \{-n+1, -n+2, \ldots, -1, 0, 1, \ldots, n-2, n-1\}$,

(ii) $h^+(A) = \max\limits_{i \in N^+} a_i$,
$h^-(A) = \max\limits_{i \in N^-} a_i$,
$h(A) = \min\{h^+(A), h^-(A)\}$,

(iii) $I^+(A) = \{i \in N^+;\, a_i \geq h(A)\}$,
$I^-(A) = \{i \in N^-;\, a_i \geq h(A)\}$,
$I(A) = \{i \in N^*;\, a_i \geq h(A)\}$,

iv) $\tilde{N}(A) = \begin{cases} N^+ & \text{if } h(A) = h^+(A), \\ N^- & \text{otherwise.} \end{cases}$

The possibility $\tilde{N}(A) = N^+$ contains a case where $h^+(A) = h^-(A)$, too.

Remark 3.1. Note that $h(A)$ is the maximal threshold level for which both sets $I^+(A), I^-(A)$ are non-empty. For any $h' > h(A)$, the threshold digraph $\mathscr{G}(A,h')$ is either trivial (there is no edge in $\mathscr{G}(A,h')$) in the case $a_0 \leq h(A)$, or all strongly connected components consist of exactly one node with the loop in the case $a_0 > h'$.

According to [2], we denote by $\mathscr{C}^*$ the following condition of a Toeplitz matrix:

$$i - j \leq n \text{ holds for every } i \in I^+(A), j \in I^-(A). \qquad (\mathscr{C}^*)$$

Theorem 3.1. [2]
Let $A = Tp(a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1}) \in B(n,n)$ be a Toeplitz matrix fulfilling condition $\mathscr{C}^*$. Then $\operatorname{per} A = \operatorname{per}\mathscr{K}$ *for any component $\mathscr{K} \in \mathrm{SCC}^{\star}\mathscr{G}(A,h(A))$.*

Remark 3.2. If a Toeplitz matrix $A \in B(n,n)$ fulfills condition $\mathscr{C}^*$, then all strongly connected components in $\mathscr{G}(A,h(A))$ are non-trivial, but not isomorphic in general.

Theorem 3.2. [2] *Let $A \in B(n,n)$ be a Toeplitz matrix fulfilling condition $\mathscr{C}^*$, let $I(A) = \{i_0, i_1, \dots, i_{k-1}\}$. Then*

$$\operatorname{per} A = \gcd\left(\frac{i_0 - i_1}{\gcd(n, i_0, i_1)}, \frac{i_0 - i_2}{\gcd(n, i_0, i_1, i_2)}, \dots \right.$$
$$\left. \dots, \frac{i_0 - i_{k-1}}{\gcd(n, i_0, \dots, , i_{k-1})}\right). \qquad (2)$$

Remark 3.3. According to [2], the period of Toeplitz matrix can be computed in $O(n)$ time.

4. ROBUSTNESS OF INTERVAL MATRICES

In this section we shall deal with matrices with interval elements. Similarly to [1], [3], [4], [8], [9], [11] we define an interval matrix $\mathbf{A}$.

Definition 4.1. *Let $\underline{A}, \overline{A} \in B(n,n)$, $\underline{A} \leq \overline{A}$. An interval matrix $\mathbf{A}$ with bounds $\underline{A}$ and $\overline{A}$ is defined as follows*

$$\mathbf{A} = [\underline{A}, \overline{A}] = \left\{A \in B(n,n); \underline{A} \leq A \leq \overline{A}\right\}.$$

Investigating the robustness of an interval matrix $\mathbf{A}$, the following questions can arise: *Is A robust for some $A \in \mathbf{A}$ or for all $A \in \mathbf{A}$?*

Definition 4.2. *An interval matrix $\mathbf{A}$ is called*

- *possibly robust if there exists a matrix $A \in \mathbf{A}$ such that A is robust,*
- *universally robust if each matrix $A \in \mathbf{A}$ is robust.*

Possible and universal robustness of general interval matrices were studied in [7].

Let us denote $H = \{\overline{a}_{ij}; i, j \in N\} \cup \{\underline{a}_{ij}; i, j \in N\}$.

Theorem 4.1. [7] *An interval matrix $\mathbf{A}$ is possibly robust if and only if for each $h \in H$ and for each $\mathscr{K} \in \mathrm{SCC}^\star \mathscr{G}(\overline{A}, h)$ such that* $\operatorname{per} \mathscr{K} \neq 1$ *the digraph $\mathscr{G}(\underline{A}, h)/V_{\mathscr{K}}$ is acyclic.*

For a given $h \in H$ let us denote $\tilde{V}_h = N \setminus \bigcup_{j=1}^{s_h} V_{\mathscr{K}_h^j}$, where $\mathscr{K}_h^1, \dots, \mathscr{K}_h^{s_h} \in \mathrm{SCC}^\star \mathscr{G}(\underline{A}, h)$.

Theorem 4.2. [7] *Let $\mathbf{A}$ be an interval matrix. Then $\mathbf{A}$ is universally robust if and only if $\underline{A}$ is robust and* $\operatorname{per}(\mathscr{G}(\underline{A}, h) \cup c) = 1$ *for each $h \in H$ and for each cycle $c \in \mathscr{G}(\overline{A}, h)/\tilde{V}_h$.*

According to [7], the complexity of checking the possible robustness of a given interval matrix is $O(n^5)$, whereby checking the universal robustness has exponentially large complexity.

5. ROBUSTNESS OF INTERVAL TOEPLITZ MATRICES

The possible and universal robustness of interval Toeplitz matrices are studied in this section. The necessary and sufficient conditions which can be checked in polynomial time are given.

Definition 5.1. *Let $\underline{A}, \overline{A}$ be Toeplitz matrices of order n such that $\underline{A} \leq \overline{A}$. An interval Toeplitz matrix $\mathbf{A}^{Tp}$ is the set of all Toeplitz matrices $A \in [\underline{A}, \overline{A}]$. We denote an interval Toeplitz matrix $\mathbf{A}^{Tp}$ by abbreviation*

$$\mathbf{A}^{Tp} = Tp\left([\underline{a}_{-n+1}, \overline{a}_{-n+1}], \dots [\underline{a}_{-1}, \overline{a}_{-1}], [\underline{a}_0, \overline{a}_0],\right.$$
$$\left.[\underline{a}_1, \overline{a}_1], \dots, [\underline{a}_{n-1}, \overline{a}_{n-1}]\right).$$

There are matrices in $\mathbf{A}$ that are not Toeplitz, so $\mathbf{A} \neq \mathbf{A}^{Tp}$. On the other hand $\underline{A}, \overline{A} \in \mathbf{A}^{Tp}$, therefore the set $\mathbf{A}^{Tp}$ is always non-empty.

5.1. Possible robustness

Definition 5.2. *We say that an interval Toeplitz matrix $\mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$ if each Toeplitz matrix $A \in \mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$.*

Let us define the Toeplitz matrix $\tilde{A} = Tp\left(\tilde{a}_{-n+1}, \dots, \tilde{a}_{-1}, \tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_{n-1}\right)$ as follows:

$$\tilde{a}_i = \begin{cases} \min\{h(\underline{A}), \overline{a}_i\} & \text{for } i \in \tilde{N}(\underline{A}), \\ \overline{a}_i & \text{for } i \notin \tilde{N}(\underline{A}). \end{cases} \qquad (3)$$

It is easy to see that $h(\tilde{A}) = h(\underline{A})$ and $\tilde{N}(\tilde{A}) = \tilde{N}(\underline{A})$.

Lemma 5.1. *Let $\mathbf{A}^{Tp}$ be an interval Toeplitz matrix and $\tilde{A}$ be the matrix defined by (3). Then $I(A) \subseteq I(\tilde{A})$ for each $A \in \mathbf{A}^{Tp}$.*

Proof. We shall prove that $I^+(A) \subseteq I^+(\tilde{A})$, $I^-(A) \subseteq I^-(\tilde{A})$ and if $0 \in I(A)$, then $0 \in I(\tilde{A})$, for each $A \in \mathbf{A}^{Tp}$.

Without any loss of generality we can suppose that $\tilde{N}(\underline{A}) = N^+$. Thus $I^+(\tilde{A}) = \{i \in N^+; \tilde{a}_i = h(\tilde{A})\} = \{i \in N^+; \overline{a}_i \geq h(\tilde{A})\}$. Let $A \in \mathbf{A}^{Tp}$ be arbitrary. We shall distinguish two cases.

Case 1. If $\tilde{N}(A) = N^+$, then $I^+(A) = \{i \in N^+; a_i = h(A)\}$ and $I^-(A) = \{i \in N^-; a_i \geq h(A)\}$. For the sets $I^-(A), I^-(\tilde{A})$ we get

$$I^-(A) = \{i \in N^-; a_i \geq h(A)\} \subseteq \{i \in N^-; \tilde{a}_i \geq h(\tilde{A})\} = I^-(\tilde{A}).$$

Further, we will prove that $I^+(A) \subseteq I^+(\tilde{A})$. Let $r \in I^+(A)$ be arbitrary. We get

$$\overline{a}_r \geq a_r = \max_{k \in N^+} a_k \geq h(\tilde{A})$$

which implies $r \in I^+(\tilde{A})$. Consequently $I^+(A) \subseteq I^+(\tilde{A})$.

Case 2. If $\tilde{N}(A) = N^-$, then

$$I^-(A) = \{i \in N^-; a_i = h(A)\} \subseteq \{i \in N^-; a_i \geq h(\tilde{A})\} \subseteq$$
$$\{i \in N^-; \tilde{a}_i \geq h(\tilde{A})\} = I^-(\tilde{A}).$$

Further, we prove that $I^+(A) \subseteq I^+(\tilde{A})$. We obtain

$$I^+(A) = \{i \in N^+; a_i \geq h(A)\} \subseteq \{i \in N^+; a_i \geq h(\tilde{A})\} \subseteq$$
$$\{i \in N^+; \overline{a}_i \geq h(\tilde{A})\} = I^+(\tilde{A}).$$

In both cases if $0 \in I(A)$, then $\tilde{a}_0 = \overline{a}_0 \geq a_0 \geq h(A) \geq h(\tilde{A})$, so $0 \in I(\tilde{A})$. □

Theorem 5.1. *An interval Toeplitz matrix $\mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$ if and only if the matrix $\tilde{A}$ fulfills condition $\mathscr{C}^*$.*

Proof. If the matrix $\tilde{A}$ fulfills condition $\mathscr{C}^*$, i.e., $i-j\le n$ for each $i\in I^+(\tilde{A})$, $j\in I^-(\tilde{A})$, then, in view of Lemma 5.1, for each $A\in\mathbf{A}^{Tp}$ the inequality $i-j\le n$ holds for each $i\in I^+(A)$, $j\in I^-(A)$. Thus an interval Toeplitz matrix fulfills condition $\mathscr{C}^*$.

The converse implication is trivial. □

Theorem 5.2. *An interval Toeplitz matrix $\mathbf{A}^{Tp}$ fulfilling condition $\mathscr{C}^*$ is possibly robust if and only if the matrix $\tilde{A}$ is robust.*

Proof. Since $I(A)\subseteq I(\tilde{A})$, according to (2) we get $\operatorname{per}\tilde{A}\,|\operatorname{per}A$. If $\tilde{A}$ is not robust, i.e., $\operatorname{per}\tilde{A}\neq 1$, then $\operatorname{per}A\neq 1$ for each $A\in\mathbf{A}^{Tp}$. Thus an interval Toeplitz matrix $\mathbf{A}^{Tp}$ is not possibly robust.

The converse implication is trivial. □

According to Theorem 5.2 checking the possible robustness of a given interval Toeplitz matrix fulfilling condition $\mathscr{C}^*$ consists of $O(n)$ arithmetic operations needed for the construction of the matrix $\tilde{A}$, $O(n)$ operations for checking whether $\tilde{A}$ fulfills condition $\mathscr{C}^*$ and $O(n)$ operations for computing $\operatorname{per}\tilde{A}$ by (2). So the complexity of checking the possible robustness of a given interval Toeplitz matrix fulfilling condition $\mathscr{C}^*$ is $O(n)$, which substantially improves the $O(n^5)$ algorithm for checking the possible robustness of an interval matrix in general case.

Example 5.1. *Let*

$$\mathbf{A}^{Tp}=T([1,2],[4,5],[2,2],[1,2],[2,4],[3,5],[1,2]).$$

We decide whether $\mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$ and in positive case we check the possible robustness of $\mathbf{A}^{Tp}$.

First, we compute $h(\underline{A})$ and construct the matrix $\tilde{A}$. We have $h^+(\underline{A})=3$, $h^-(\underline{A})=4$ and $h(\underline{A})=3$, which implies $\tilde{N}(\underline{A})=N^+$. By (3), we get $\tilde{A}=T(2,5,2,2,3,3,2)$. We have $h(\tilde{A})=3$ and $I(\tilde{A})=\{-2,1,2\}$. Since $i-j\le 4=n$ for each $i\in I^+(\tilde{A})$, $j\in I^-(\tilde{A})$, the matrix $\tilde{A}$ fulfills condition $\mathscr{C}^*$. According to Theorem 5.1, $\mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$.

Further, we check the possible robustness of $\mathbf{A}^{Tp}$. We compute $\operatorname{per}\tilde{A}$ by (2):

$$\operatorname{per}\tilde{A}=\gcd\Big(\frac{-3}{\gcd(4,-2,1)},\frac{-4}{\gcd(4,-2,1,2)}\Big)=1,$$

so the matrix $\tilde{A}$ is robust. In view of Theorem 5.2, the given interval Toeplitz matrix is possibly robust.

5.2. Universal robustness

Theorem 5.3. *Let $\mathbf{A}^{Tp}$ be an interval Toeplitz matrix fulfilling condition $\mathscr{C}^*$. If $h(\underline{A})=h(\overline{A})$ and the matrix $\underline{A}$ is robust, then $\mathbf{A}^{Tp}$ is universally robust.*

Proof. Let $\mathbf{A}^{Tp}$ be an interval Toeplitz matrix fulfilling condition $\mathscr{C}^*$. Suppose that $\mathbf{A}^{Tp}$ is not universally robust and the matrix $\underline{A}$ is robust. If $A\in\mathbf{A}^{Tp}$ is not robust, then by Theorem 3.1 we obtain $\operatorname{per}\tilde{\mathscr{K}}=\operatorname{per}A\neq 1$ for each $\tilde{\mathscr{K}}\in\mathrm{SCC}^\star\mathscr{G}(A,h(A))$. In view of Remark 3.2 and Lemma 2.2, for each $h\le h(\underline{A})$ and for each $\mathscr{K}\in\mathrm{SCC}^\star\mathscr{G}(A,h)$ there exists $\mathscr{K}'\in\mathrm{SCC}^\star\mathscr{G}(\underline{A},h(\underline{A}))$ such that $\mathscr{K}'\subseteq\mathscr{K}$.

The robustness of $\underline{A}$ implies that $\operatorname{per}\mathscr{K}'=1$ for each $\mathscr{K}'\in\mathrm{SCC}^\star\mathscr{G}(\underline{A},h(\underline{A}))$. According to Lemma 2.1 we get $\operatorname{per}\mathscr{K}=1$ for each $\mathscr{K}\in\mathrm{SCC}^\star\mathscr{G}(A,h)$ and for each $h\le h(A)$. Then the existence of $\tilde{\mathscr{K}}\in\mathrm{SCC}^\star\mathscr{G}(A,h(A))$ such that $\operatorname{per}\tilde{\mathscr{K}}\neq 1$ implies $h(A)>h(\underline{A})$. Hence $h(\underline{A})<h(\overline{A})$. □

Example 5.2. *Let*

$$\mathbf{A}^{Tp}=T([1,2],[4,5],[3,3],[1,2],[1,3],[3,3],[1,2]).$$

We decide whether $\mathbf{A}^{Tp}$ is universally robust.

Since $\tilde{A}=T(2,5,3,2,3,3,2)$, $\tilde{A}$ fulfills condition $\mathscr{C}^*$ and by Theorem 5.1 $\mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$. Since $h(\underline{A})=h(\overline{A})=3$ and $\operatorname{per}\underline{A}=1$, in view of Theorem 5.3 a given interval Toeplitz matrix is universally robust.

Theorem 5.3 represents a sufficient, but not necessary condition for the universal robustness of an interval Toeplitz matrix.

For each $k\in N^+$, $l\in N^-$ we define the Toeplitz matrix $A^{(kl)}=Tp\,(a^{(kl)}_{-n+1},\dots,a^{(kl)}_{-1},a^{(kl)}_0,a^{(kl)}_1,\dots,a^{(kl)}_{n-1})$ as follows:

$$a^{(kl)}_i=\begin{cases}\overline{a}_i & \text{for } i\in\{k,l\},\\ \underline{a}_i & \text{otherwise.}\end{cases}\tag{4}$$

Theorem 5.4. *Let $\mathbf{A}^{Tp}$ be an interval Toeplitz matrix fulfilling condition $\mathscr{C}^*$. $\mathbf{A}^{Tp}$ is universally robust if and only if $\underline{A}$ is robust and for each $k\in N^+$, $l\in N^-$ such that $\min\{\overline{a}_k,\overline{a}_l\}>h(\underline{A})$ the matrix $A^{(kl)}$ is robust.*

Proof. If $\underline{A}$ is not robust or there exist $k\in N^+$, $l\in N^-$ such that $\min\{\overline{a}_k,\overline{a}_l\}>h(\underline{A})$ and the matrix $A^{(kl)}$ is not robust, then $\mathbf{A}^{Tp}$ is not universally robust.

For the converse implication suppose that $\mathbf{A}^{Tp}$ is not universally robust and $\underline{A}$ is robust. We will prove that there exist $k\in N^+$, $l\in N^-$ such that $\min\{\overline{a}_k,\overline{a}_l\}>h(\underline{A})$ and the matrix $A^{(kl)}$ is not robust.

Let $A\in\mathbf{A}^{Tp}$ be such that $\operatorname{per}A\neq 1$, i.e., $\operatorname{per}\mathscr{K}'\neq 1$, where $\mathscr{K}'\in\mathrm{SCC}^\star\mathscr{G}(A,h(A))$. Similarly as in the proof of Theorem 5.3, the robustness of $\underline{A}$ implies $h(A)>h(\underline{A})$. Let $k\in N^+$, $l\in N^-$ be such that $h^+(A)=a_k$, $h^-(A)=a_l$. Since $h^+(A^{(kl)})=\overline{a}_k$ and $h^-(A^{(kl)})=\overline{a}_l$ we obtain $h(A^{(kl)})=\min\{\overline{a}_k,\overline{a}_l\}\ge h(A)>h(\underline{A})$.

Let $j\in N^*$ be such that $j\notin I(A)$. Then $a^{(kl)}_j=\underline{a}_j\le a_j<h(A)\le h(A^{(kl)})$. Hence $j\notin I(A^{(kl)})$. Thus $I(A^{(kl)})\subseteq I(A)$. According to (2), we obtain $\operatorname{per}A\mid\operatorname{per}A^{(kl)}$ which implies $\operatorname{per}A^{(kl)}\neq 1$. Thus the matrix $A^{(kl)}$ is not robust. □

According to Theorem 5.4 checking the universal robustness of a given interval Toeplitz matrix fulfilling condition $\mathscr{C}^*$ consists of $O(n)$ arithmetic operations needed for checking whether $\mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$, $O(n)$ operations for checking the robustness of $\underline{A}$ by (2) and at most $n^2O(n)=O(n^3)$ operations for checking the robustness of matrices $A^{(kl)}$. So the complexity of checking the universal robustness of a given interval Toeplitz matrix fulfilling condition $\mathscr{C}^*$ is $O(n^3)$, which substantially improves the exponential algorithm for checking the universal robustness of an interval matrix in general case.

Example 5.3. *We decide whether* $\mathbf{A}^{Tp}$ *is universally robust, if*

$$\mathbf{A}^{Tp} = T([1,2],[4,5],[2,2],[1,2],[3,4],[3,5],[1,2]).$$

Since $\tilde{A} = T(2,5,2,2,3,3,2)$, $\tilde{A}$ fulfills condition $\mathscr{C}^*$ and by Theorem 5.1 the given interval Toeplitz matrix $\mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$. We have $h(\underline{A}) = 3$ and by (2) we get per$\underline{A} = 1$, so $\underline{A}$ is robust. Since $h(\overline{A}) = 5 \neq h(\underline{A})$, the sufficient condition from Theorem 5.3 is not satisfied, so we shall continue with checking the condition from Theorem 5.4.

For $k \in N^+$, the inequality $\overline{a}_k > 3$ is fulfilled for $k \in \{1,2\}$ and for $l \in N^-$, the inequality $\overline{a}_l > 3$ is fulfilled for $l = -2$. We will construct the corresponding matrices $A^{(kl)}$ by (4).

For $k = 1$, $l = -2$ we get $A^{(kl)} = T(1,5,2,1,4,3,1)$. We have $I(A^{(kl)}) = \{-2,1\}$ and by (2) we get per $A^{(kl)} = 3$.

In view of Theorem 5.4 the given interval Toeplitz matrix is not universally robust.

Example 5.4. *Let*

$$\mathbf{A}^{Tp} = T([2,3],[6,6],[6,6],[1,2],[4,8],[3,4],[1,3]).$$

Decide whether $\mathbf{A}^{Tp}$ *is universally robust.*

Since $\tilde{A} = T(3,6,6,2,4,4,3)$, $\mathbf{A}^{Tp}$ fulfills condition $\mathscr{C}^*$. We have $h(\underline{A}) = 4$ and by (2) we get per$\underline{A} = 1$, so $\underline{A}$ is robust. Since $h(\overline{A}) = 6 \neq h(\underline{A})$, we have to check the condition from Theorem 5.4.

For $k \in N^+$, the inequality $\overline{a}_k > 4$ is fulfilled for $k = 1$ and for $l \in N^-$, the inequality $\overline{a}_l > 4$ is fulfilled for $l \in \{-2,-1\}$.

For $k = 1$, $l = -2$ we get $A^{(kl)} = T(2,6,6,1,8,3,1)$. We have $I(A^{(kl)}) = \{-2,-1,1\}$. By (2) we get per $A^{(kl)} = 1$.

For $k = 1$, $l = -1$ we obtain $A^{(kl)} = T(2,6,6,1,8,3,1)$ which is identical to the matrix from the previous case, so $A^{(kl)}$ is robust.

According to Theorem 5.4 the given interval Toeplitz matrix is universally robust.

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