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# ON THE OPTIMAL DRAWINGS OF PRODUCTS OF PATHS WITH GRAPHS

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**ABSTRACT**

*There are known several exact results of the crossing numbers of the Cartesian product of all graphs of order at most four with paths, cycles and stars. Moreover, for the path $P_n$ of length n, the crossing numbers of Cartesian products $G\square P_n$ for all connected graphs G on five vertices and for forty graphs G on six vertices are given. In the paper, we extend these results by determining the crossing numbers for the Cartesian products of paths with two special graphs of order six.*

***Keywords:*** *Cartesian product, crossing number, drawing, graph, path*

## 1. INTRODUCTION

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A drawing of a graph is a mapping of a graph into a surface. The crossing number $\mathrm{cr}(G)$ of a simple graph $G$ is defined as the minimum possible number of edge crossings in a drawing of $G$ in the plane. A survey of the other variants of crossing numbers can be found in [12]. A drawing with minimum number of crossings (an optimal drawing) must be a good drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing. Moreover, no three edges cross in a point. The investigation on the crossing number of graphs is very difficult problem. It is well known that this problem is NP-complete. The crossing numbers has been studied to improve the readability of hierarchical structures. A crossing of two edges of the communication graph requires unit area in VLSI-layout. So, the crossing number together with the number of vertices of the graph immediately provide a lower bound for the area of the VLSI-layout of the communication graph. For that reason the problem of crossing numbers was studied also by VLSI communities and computer scientists.

According to their special structure, Cartesian products of special graphs are one of few graph classes for which the exact values of crossing numbers were obtained. The Cartesian product $G_1\square G_2$ of graphs $G_1$ and $G_2$ has vertex set $V(G_1\square G_2) = V(G_1) \times V(G_2)$ and any two vertices $(u,u')$ and $(v,v')$ are adjacent in $G_1\square G_2$ if and only if either $u = v$ and $u'$ is adjacent with $v'$ in $G_2$, or $u' = v'$ and $u$ is adjacent with $v$ in $G_1$.

Let $C_n$ be the cycle of length $n$, $P_n$ be the path of length $n$, and $S_n$ be the star isomorphic to $K_{1,n}$. Beineke and Ringeisen [1] started to study the crossing numbers of Cartesian products of cycles with all graphs on at most four vertices. The crossing numbers of Cartesian products of cycles, paths and stars with all 4-vertex graphs are determined in [3], [4], and [5]. The crossing numbers of Cartesian products of paths with all graphs of order five are collected in [8]. It seems natural to enquire about crossing numbers of Cartesian products of paths with other graphs. There are known the crossing numbers of products $G\square P_n$ for some 6-vertex graphs $G$, see [6], [10], [11], [13], [14], and [15]. The crossing numbers of Cartesian products of paths with 40 graphs of order six are collected in [9]. In this paper, we extend these results by determining the crossing numbers of the Cartesian products of two special 6-vertex graphs $F$ and $H$ shown in Fig. 1 with the path $P_n$.

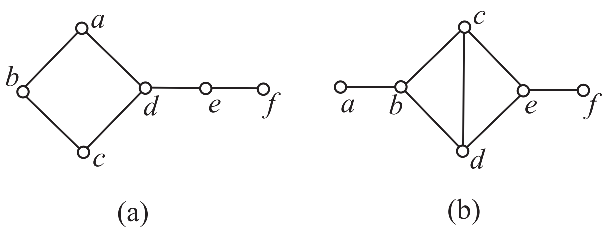


**Fig. 1** The graphs $F$ and $H$ of order six.

Let $D$ be a good drawing of the graph $G$. We denote the number of crossings in $D$ by $\mathrm{cr}_D(G)$. Let $G_i$ and $G_j$ be edge-disjoint subgraphs of $G$. We denote by $\mathrm{cr}_D(G_i, G_j)$ the number of crossings between edges of $G_i$ and edges of $G_j$. In a good drawing $D$ of the graph $G$, we say that a cycle $C$ separates the cycles $C'$ and $C''$ (the vertices of a subgraph $G$ not containing vertices of $C$) if $C'$ and $C''$ (the vertices of $G$) are contained in different components of $\mathbb{R}^2 \setminus C$. In the proofs of the paper, we will often use the term "region" also in non-planar drawings. In this case, the vertices are considered to be vertices in the "map".

## 2. THE GRAPHS $F\square P_N$ AND $H\square P_N$

We assume $n \geq 1$ and find it convenient to consider the graph $F\square P_n$ in the following way: it has $6(n+1)$ vertices and edges that are the edges in $n+1$ copies of $F^i$, $i = 0,1,\ldots,n$, and in six paths of length $n$, see Fig. 2. For $i = 0,1,\ldots,n$, let $a_i$, $b_i$, $c_i$, and $e_i$ be the vertices of $F^i$ of degree two, $d_i$ the vertex of degree three, and $f_i$ the vertex of degree one (see Fig. 1(a)). Let us denote by $M_F^i$ the subgraph of $F\square P_n$ containing the vertices of $F^{i-1}$ and $F^i$ and six edges joining $F^{i-1}$ to $F^i$, $i = 1,2,\ldots,n$. Let $Q_F^i$, $i = 1,2,\ldots,n-1$, denote the subgraph of $F\square P_n$ induced by $V(F^{i-1}) \cup V(F^i) \cup V(F^{i+1})$. So, $Q_F^i = F^{i-1} \cup M_F^i \cup F^i \cup M_F^{i+1} \cup F^{i+1}$.

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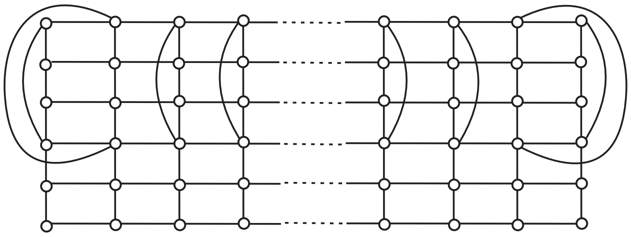

**Fig. 2** The drawing of the graph $F\Box P_n$ with $2(n-2)$ crossings.

Similarly, the graph $H\Box P_n$ has $6(n+1)$ vertices and edges that are the edges in $n+1$ copies of $H^i$ and six paths of length $n$, see Fig. 3. The vertices of $H^i$ of degree three are denoted by $b_i, c_i, d_i$, and $e_i$. The vertices of degree one adjacent to $b_i$ and $e_i$ are denoted by $a_i$, and $f_i$, respectively (see Fig. 1(b)). For $i = 1,2,\dots,n$, let $M_H^i$ denote the subgraph of $H\Box P_n$ consisting of the vertices in $H^{i-1}$ and $H^i$ and of the edges joining $H^{i-1}$ with $H^i$, and let $Q_H^i = H^{i-1} \cup M_H^i \cup H^i \cup M_H^{i+1} \cup H^{i+1}$.

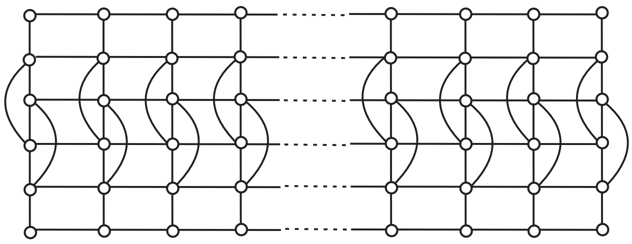

**Fig. 3** The drawing of the graph $H\Box P_n$ with $2n$ crossings.

Both graphs $F\Box P_n$ and $H\Box P_n$ contain $C_4\Box P_n$ as a subgraph. For $i = 0,1,\dots,n$, let $C_4^i$ denote the 4-cycle of the subgraph $C_4\Box P_n$ and let $M_C^i$ denote the corresponding subgraph of $M_F^i$ or $M_H^i$. The subgraph of $H$ induced on the vertices $b,c,d$, and $e$ is isomorphic with the complete tripartite graph $K_{1,1,2}$. Hence, the graph $H\Box P_n$ contains $K_{1,1,2}\Box P_n$ as a subgraph. In the graph $H\Box P_n$, let $K^i$ denote the subgraph of $H^i$ which is isomorphic with $K_{1,1,2}$, $i = 0,1,\dots,n$. By $M_K^i$ we will denote the corresponding subgraph of $M_H^i$ in $K_{1,1,2}\Box P_n$.

Consider the graph $C_4\Box P_2$ which is a subgraph of $Q_F^i$ in $F\Box P_n$ as well as a subgraph of $Q_H^i$ in $H\Box P_n$, $i = 1,2,\dots,n$. The following result enables us to simplify the proofs in the next sections.

**Lemma 2.1.** *Let D be a good drawing of the graph $C_4\Box P_2$ in which the 4-cycles $C_4^0$, $C_4^1$, and $C_4^2$ do not cross each other and none of them separates two other. Then* $\mathrm{cr}_D(C_4^1) + \mathrm{cr}_D(C_4^1, M_C^1 \cup M_C^2) + \mathrm{cr}_D(C_4^0 \cup M_C^1, C_4^2 \cup M_C^2) \geq 2$.

*Proof.* Assume that there is a good drawing $D$ of the graph $C_4\Box P_2$ in which two different 4-cycles do not cross each other and none of the 4-cycles $C_4^0$, $C_4^1$, and $C_4^2$ separates two other and that $\mathrm{cr}_D(C_4^1) + \mathrm{cr}_D(C_4^1, M_C^1 \cup M_C^2) + \mathrm{cr}_D(C_4^0 \cup M_C^1, C_4^2 \cup M_C^2) \leq 1$. Hence, in such a drawing, at least one of the subgraphs $C_4^0 \cup M_C^1$ and $C_4^2 \cup M_C^2$ does not cross $C_4^1$. Without loss of generality, let $\mathrm{cr}_D(C_4^1, C_4^0 \cup M_C^1) = 0$ and let both $C_4^0$ and $C_4^2$ are placed outside $C_4^1$. Then, regardless of the edges of $C_4^1$ cross each other or not, the subdrawing of $C_4^0 \cup M_C^1 \cup C_4^1$ induced by $D$ divides the plane in such a way that on the boundary of every region outside $C_4^1$ there are at most two vertices of $C_4^1$. The 4-cycle $C_4^2$ does not cross an edge of the 2-connected subgraph $C_4^0 \cup M_C^1 \cup C_4^1$, otherwise $\mathrm{cr}_D(C_4^2, C_4^0 \cup M_C^1 \cup C_4^1) \geq 2$, a contradiction. Thus, $C_4^2$ is placed in one region outside $C_4^1$. But, in this case, at least two edges of $M_C^2$ joining $C_4^2$ with the vertices of $C_4^1$ cross the edges of $C_4^0 \cup M_C^1 \cup C_4^1$. This contradiction completes the proof. □

## 3. THE CROSSING NUMBER OF $F\Box P_N$

The graph $F\Box P_1$ is planar. The crossing number of the graph $F\Box P_2$ is one, because the graph $S_3\Box P_2$ is its subgraph and $\mathrm{cr}(S_3\Box P_2) = 1$ (see [3]). The reverse inequality $\mathrm{cr}(F\Box P_2) \leq 1$ one can verify by finding a suitable drawing of the graph $F\Box P_2$ with one crossing. In Fig. 2 there is the drawing of the graph $F\Box P_n$ with $2(n-2)$ crossings. The next result is fundamental in proving that the crossing number of the graph $F\Box P_n$ is $2(n-2)$ for $n \geq 3$.

**Lemma 3.1.** *If D is a good drawing of the graph $F\Box P_n$, $n \geq 3$, in which every of the subgraphs $F^i$, $i = 0,1,2,...,n$, has at most one crossing on its edges, then in D there are at least $2(n-2)$ crossings.*

*Proof.* In a drawing of the graph $F\Box P_n$, let us consider the following types of possible crossings on the edges of $Q_F^i$ for all $i = 1,2,...,n-1$:

(1) a crossing of an edge in $F^{i-1} \cup M_F^i$ with an edge in $F^{i+1} \cup M_F^{i+1}$,

(2) a crossing of an edge in $M_F^i \cup M_F^{i+1}$ with an edge in $F^i$,

(3) a crossing among the edges of $F^i$.

It is readily seen that every of the considered crossings appears in a good drawing of the graph $F\Box P_n$ only on the edges of one subgraph $Q_F^i$. In a good drawing of $F\Box P_n$, we define the force $f(Q_F^i)$ of $Q_F^i$ in the following way: every crossing of type (1), (2), and (3) contributes the value 1 to $f(Q_F^i)$. The total force of the drawing is the sum of $f(Q_F^i)$. It is easy to see that the number of crossings in the drawing is not less than the total force of the drawing.

Consider now the good drawing $D$ of $F\Box P_n$ assumed in Lemma 3.1. Clearly, two 4-cycles $C_4^i$ and $C_4^j$, $i \neq j$, do not cross each other in $D$, otherwise both $F^i$ and $F^j$ are crossed at least twice. In every subdrawing $D(Q_F^i)$ of the subgraph $Q_F^i$ induced by $D$, $i = 1,2,\dots,n-1$, none of the 4-cycles $C_4^{i-1}$ and $C_4^{i+1}$ separates two other. If otherwise, $C_4^{i-1}$ separates $C_4^i$ and $C_4^{i+1}$ ($C_4^{i+1}$ separates $C_4^i$ and $C_4^{i-1}$), then the cycle $C_4^{i-1}$ ($C_4^{i+1}$) is crossed by all four edges joining the separated 4-cycles. For $i = 2,3\dots,n-2$, if $C_4^i$ separates $C_4^{i-1}$ and $C_4^{i+1}$, both paths $d_{i-1}e_{i-1}e_ie_{i+1}d_{i+1}$ and $d_{i-1}d_{i-2}e_{i-2}f_{i-2}f_{i-1}f_if_{i+1}f_{i+2}e_{i+2}d_{i+2}d_{i+1}$ cross the cycle $C_4^i$. This contradicts the assumption that every subgraph $F^i$ has at most one crossing on its edges. So, by Lemma 2.1, every subdrawing $D(Q_F^i)$, $i = 2,3,\dots,n-2$, contains at least two crossings of types (1), (2) or (3). This forces that, in $D$, there are at least $2(n-3)$ crossings among the edges of the subgraph $F^1 \cup M_F^1 \cup F^2 \cup \cdots \cup F^{n-2} \cup M_F^{n-1} \cup F^{n-1}$. Moreover, if $C_4^1$ does not separate the cycles $C_4^0$ and $C_4^2$ or if $C_4^{n-1}$ does not separate the cycles $C_4^{n-2}$ and $C_4^n$, in $D$ there are at least $2(n-2)$ crossings and we are done.

It remains to prove that if both $C_4^1$ in $Q_F^1$ and $C_4^{n-1}$ in $Q_F^{n-1}$ separate the remaining two 4-cycles, then in $D$ there are at least two crossings not counted in $\sum_{i=2}^{n-2} f(Q_F^i)$. If the cycle $C_4^1$ separates $C_4^0$ and $C_4^2$, a possible crossing between an edge of $C_4^1$ and an edge of $F^0$ or $F^2$ is not a crossing in $\sum_{i=2}^{n-2} f(Q_F^i)$. Otherwise if $\mathrm{cr}_D(C_4^1, F^0) = \mathrm{cr}_D(C_4^1, F^2) = 0$, both path $e_o e_1 e_2$ and $f_0 f_1 f_2$ cross $C_4^1$, which contradicts the restriction of at most one crossing on $C_4^1$. The same holds for the subgraph $Q_F^{n-1}$ if $C_4^{n-1}$ separates $C_4^{n-2}$ and $C_4^n$. Hence, in $D$ there are at least $\sum_{i=2}^{n-2} f(Q_F^i) + 2 = 2(n-3) + 2 = 2(n-2)$ crossings. This completes the proof. □

For the crossing number of the graph $F \square P_n$ we have the next result.

**Theorem 3.1.** $\mathrm{cr}(F \square P_n) = 2(n-2)$ *for* $n \geq 3$.

*Proof.* The drawing in Fig. 2 with $2(n-2)$ crossings confirms that $\mathrm{cr}(F \square P_n) \leq 2(n-2)$ for $n \geq 3$. We prove the reverse inequality by induction on $n$. The graph $S_3 \square P_3$ is a subgraph of $F \square P_3$ and we know that $\mathrm{cr}(S_3 \square P_3) = 2$ (see [3]). Thus, the crossing number of $F \square P_3$ is at least two and the result is true for $n = 3$. Assume that it is true for $n = k$, $k \geq 3$, and suppose that there is a good drawing of the graph $F \square P_{k+1}$ with fewer than $2(k-1)$ crossings. By Lemma 3.1, some of the subgraphs $F^i$, $i = 0, 1, ..., k+1$, must be crossed at least twice. If $F^0$ has at least two crossings on its edges, then deleting of all vertices of $F^0$ results in a drawing of the graph $F \square P_k$ with fewer than $2(k-2)$ crossings. This contradicts the induction hypothesis. The same contradiction is obtained if at least two crossings appear on the edges of $F^{k+1}$. If some $F^i$, $i \in \{1, 2, ..., k\}$, is crossed at least twice, by the removal of all edges of this $F^i$, a subdivision of $F \square P_k$ with fewer than $2(k-2)$ crossings is obtained. This contradiction with the induction hypothesis completes the proof. □

## 4. THE CROSSING NUMBER OF $H \square P_N$

In this section, the crossing number of the Cartesian product $H \square P_n$ described in Section 2 is given. It is easy to see that the graph $H \square P_1$ is planar and the next lemma determines the crossing number of the graph $H \square P_2$.

**Lemma 4.1.** $\mathrm{cr}(H \square P_2) = 4$.

*Proof.* The graph $H \square P_2$ is isomorphic with the subgraph $Q_H^1$ defined above. Hence, $H \square P_2 = H^0 \cup M_H^1 \cup H^1 \cup M_H^2 \cup H^2$. It can be easily seen in Fig. 3 that the inequality $\mathrm{cr}(H \square P_2) \leq 4$ holds. To prove the inverse inequality, assume that there is a good drawing of the graph $H \square P_2$ with less than four crossings and let $D$ be such a drawing. In $D$, none of $K^i$, $i \in \{0, 1, 2\}$, cross both subgraphs $K^j$ and $K^l$, $j, l \in \{0, 1, 2\}$, $j, l \neq i$, $j \neq l$, because if two different graphs both isomorphic with $K_{1,1,2}$ cross each other in a good drawing, then they cross at least twice. We will show that, in $D$, two different subgraphs $K^i$ and $K^j$, $i, j \in \{0, 1, 2\}$, do not cross each other and that none of the subgraphs $K^0, K^1$, and $K^2$ separates two others.

Assume first that $\mathrm{cr}_D(K^0, K^2) \neq 0$. As the drawing $D$ is good, at least one of $K^0$ and $K^2$ separates the vertices of the other. Without loss of generality, let $K^0$ separates the vertices of $K^2$. In such a case, $K^0$ separates at least one vertex of $K^2$ from the vertices of $K^1$ and therefore, at least one edge of $M_K^2$ crosses the edges of $K^0$. The restriction of at most three crossings in $D$ forces that $\mathrm{cr}_D(H^0, H^2) = 2$. It is easy to see that, in this case, $K^0$ separates one of the edges $a_2 b_2$ and $e_2 f_2$ from the subgraph $H^1$ and that the edges of $M_H^2 \cup H^1$ cross the edges of $K^0$ at least twice. This contradicts the assumption of at most three crossings in $D$ and therefore, $\mathrm{cr}_D(K^0, K^2) = 0$. Now, without loss of generality, assume that $\mathrm{cr}_D(K^0, K^1) \neq 0$. If $K^0$ separates the vertices of $K^1$, then the similar consideration as above shows that there are more than three crossings in $D$, a contradiction. The last case to consider is that $K^1$ separates the vertices of $K^0$. If $\mathrm{cr}_D(K^0, K^1) = 3$, then no crossing can appear in the subdrawing of $K^1 \cup M_K^2 \cup K^2$ induced by $D$ and the subdrawing $D(K^1 \cup M_K^2 \cup K^2)$ divides the plane as shown in Fig. 4(a). Then, in $D$, the path $b_0 a_0 a_1 a_2 b_2$ does not cross the edges of $D(K^1 \cup M_K^2 \cup K^2)$ and $K^0$ must be placed in two neighbouring regions of the subdrawing $D(K^1 \cup M_K^2 \cup K^2)$ bounded by the edge $b_1 c_1$ or by the edge $b_1 d_1$. In both cases, the path $e_0 f_0 f_1 f_2 e_2$ crosses the edges of $K^1 \cup M_K^1 \cup K^2$ at least once, a contradiction. Thus, $\mathrm{cr}_D(K^0, K^1) = 2$ and $K^1$ separates the edges $a_0 b_0$ and $e_0 f_0$. If, in this case, $K^1$ separates $K^0$ and $K^2$, both paths $b_0 a_0 a_1 a_2 b_2$ and $e_0 f_0 f_1 f_2 e_2$ cross the edges of $K^1$ and in $D$ there are more than three crossings. Hence, the unique possibility is that one of the edges $a_0 b_0$ and $e_0 f_0$, say $a_0 b_0$, is separated from the vertices of $K^2$. This forces one other crossing between the edges of $K^1$ and the path $a_0 a_1 a_2$. So, the subdrawing $D(K^1 \cup M_K^1 \cup K^2)$ without crossings is the same as in Fig. 4(a). Now, in $D$, the vertices of $K^0$ are placed in two regions of the subdrawing $D(K^1 \cup M_K^2 \cup K^2)$ with at most three vertices of $K^1$ on their boundaries and therefore, at least one crossing must appear between the edges of $M_K^1$ and the edges of $K^1 \cup M_K^2 \cup K^2$. This contradiction confirms that $\mathrm{cr}_D(K^i, K^j) = 0$ for $i \neq j$, $i, j \in \{0, 1, 2\}$.

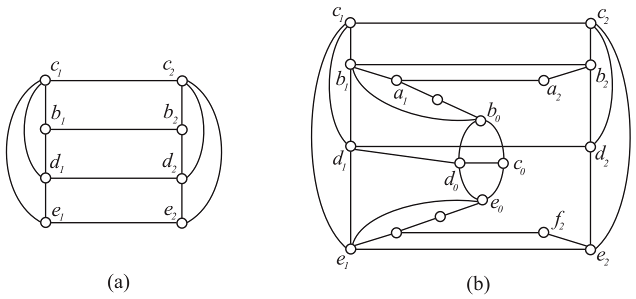


**Fig. 4** The subdrawing of $K^1 \cup M_K^2 \cup K^2$ without crossings and the drawing of $H \square P_2$ without the edge $c_0 c_1$ with only two crossings.

The subgraph $K^0$ ($K^2$) does not separate $K^1$ and $K^2$ ($K^0$ and $K^1$), otherwise at least four edges of $M_K^2$ ($M_K^1$) cross $K^0$

$(K^2)$. Assume now that $K^1$ separates $K^0$ and $K^2$. Thus, in $D$, both paths $b_0a_0a_1a_2b_2$ and $e_0f_0f_1f_2e_2$ cross $K^1$. This forces that in the subdrawing of $C_4^1 \cup M_C^2 \cup C_4^2$ induced by $D$ there is at most one crossing. It is well known that there is no good drawing of $C_4\Box P_1$ with exactly one crossing, because for any two edges which cross each other one can find two vertex-disjoint cycles such that crossed edges are in different cycles. Two vertex-disjoint cycles cannot cross only once. Hence, the subdrawing of $C_4^1 \cup M_C^2 \cup C_4^2$ without crossings one can obtain by deleting the edges $c_1d_1$ and $c_2d_2$ from the drawing in Fig. 4(a). If, in $D$, the edge $c_1d_1$ does not cross an edge of $M_C^2$, then it is placed as shown in Fig. 4(a) and $K^0$ is placed in one of two triangular regions inside $K^1$. Due to symmetry, let $K^0$ is placed in the region with the vertices $c_1, d_1$, and $e_1$ on its boundary. So, in $D$, the path $b_0a_0a_1a_2b_2$ crosses $K^1 \cup M_K^2 \cup K^2$ at least twice, because the vertices $b_0$ and $b_2$ are separated by two edge-disjoint cycles $c_1d_1e_1c_1$ and $c_1b_1d_1d_2e_2c_2c_1$. Moreover, $K^1$ is crossed by the edge $b_0b_1$, too. This contradicts the assumption of at most three crossings in $D$. The last possibility is that the edge $c_1d_1$ outside $C_4^1$ crosses one of the edges $b_1b_2$ and $e_1e_2$. Without loss of generality, let the edge $c_1d_1$ crosses the edge $b_1b_2$. In this case, the edge $c_2d_2$ must be placed inside $C_4^2$ and the vertices $b_0$ and $b_2$ are separated by the edge-disjoint cycles $c_1b_1d_1e_1c_1$ and $c_1d_1d_2c_2c_1$, which forces that the path $b_0a_0a_1a_2b_2$ crosses the edges of $K^1 \cup M_K^2 \cup K^2$ at least twice and in $D$ there are at least four crossings again. So, none of the subgraphs $K^0, K^1$, and $K^2$ separates two others in the considered drawing $D$.

Now we show that, in $D$, no edge of $M_K^1$ crosses $K^2$ as well as no edge of $M_K^2$ crosses $K^0$. Without loss of generality let an edge of $M_K^2$ crosses $K^0$. As two different $K^i$ and $K^j$ do not cross, $\mathrm{cr}_D(K^0, M_K^2) \geq 2$. As there is no good drawing of the graph $C_4\Box P_1$ with only one crossing, the subdrawing of $D$ induced by the subgraph $C_4^1 \cup M_C^2 \cup C_4^2$ without crossings divides the plane in such a way that at most two vertices of $K^1$ are on the boundary of every region outside $K^1$. Since $\mathrm{cr}_D(K^0, M_K^2) \leq 3$, only one edge of $M_K^2$ crosses $K^0$ and therefore, at least one vertex of $K^1$ is not on the boundaries of the regions with the vertices of $K^0$ inside. This requires at least one crossing between an edge of $M_K^1$ and an edge of $K^1 \cup M_K^2 \cup K^2$. The drawing $D$ is assumed with at most three crossings and therefore, the unique subdrawing of $K^1 \cup M_K^2 \cup K^2$ induced by $D$ is shown in Fig. 4(a). None of the edges $b_1b_2$ and $e_1e_2$ crosses $K^0$, otherwise, in $D$, at least two paths joining $e_0$ to $e_1$ or $b_0$ to $b_1$ cross the edges of $K^1 \cup M_K^2 \cup K^2$, a contradiction. Due to symmetry, let the edge $d_1d_2$ crosses $K^0$. Hence, $K^0$ is placed into two regions of the subdrawing $D(K^1 \cup M_K^2 \cup K^2)$ in such a way that the edge $c_0c_1$ crosses the cycle $b_1b_2d_2e_2e_1d_1b_1$ and therefore, the edges $b_0b_1$ and $e_0e_1$ do not cross the edges of $K^1 \cup M_K^2 \cup K^2$. This forces that the edges of $K^0$ do not cross each other and that both edges $b_0c_0$ and $b_0d_0$ or both edges $e_0c_0$ and $e_0d_0$ cross the edge $d_1d_2$. Without loss of generality assume the first case. Since the edge $d_0d_1$ as well as the paths $b_0a_0a_1b_1$, $e_0f_0f_1e_1$, $a_1a_2b_2$, and $f_1f_2e_2$ are not crossed, the necessary subdrawing of $D$ obtained by deleting the edge $c_0c_1$ is shown in Fig. 4(b). One can easy to verify that it is impossible to add the edge $c_0c_1$ with only one crossing. This confirms that $\mathrm{cr}_D(K^0, M_K^2) = \mathrm{cr}_D(K^2, M_K^1) = 0$.

In this paragraph we show that in $D(K^0 \cup M_K^1 \cup K^1 \cup M_K^2 \cup K^2)$ there are at least two crossings other than the crossings among the edges of $K^0 \cup M_K^1$ as well as the crossings among the edges of $M_K^2 \cup K^2$. Needless to say, two different $K^i$ and $K^j$ do not cross. If both $M_K^1$ and $M_K^2$ cross $K^1$, we are done. Otherwise, without loss of generality, let $\mathrm{cr}_D(K^1, M_K^1) = 0$. Assume now the subdrawing of $K^0 \cup M_K^1 \cup K^1$. Regardless of the edges of $K^1$ cross each other or not, $D(K^0 \cup M_K^1 \cup K^1)$ divides the plane in such a way that on the boundary of every region outside $K^1$ there are at most two vertices of $K^1$, see the drawings in Fig. 5, where possible crossings among the edges of $K^0 \cup M_K^1$ are inside the dotted cycle. As $\mathrm{cr}_D(K^2, K^0 \cup M_K^1 \cup K^1) = 0$, it is easy to verify that every placing of the subgraph $K^2$ outside $K^1$ enforces at least two crossings between the edges of $M_K^2$ and the edges of $K^0 \cup M_K^1 \cup K^1$.

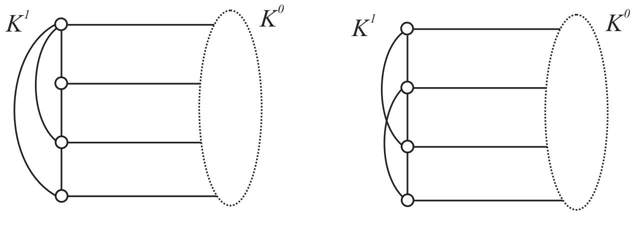


**Fig. 5** The subdrawings of $K^0 \cup M_K^1 \cup K^1$ without crossings between $K^0 \cup M_K^1$ and $K^1$.

In the rest of the proof we show that in $D$ there are at least two crossings other than the two crossings considered in the previous paragraph. Consider now the subgraph of $Q_H^1$ consisting of $K^0$, the edge $c_0c_1$, the paths $c_1b_1b_0$, $c_1d_1d_0$, $c_1e_1e_0$, and the path $b_0a_0a_1a_2b_2c_2d_2e_2f_2f_1f_0e_0$, and let us denote it by $K_5^s(K^0)$, see the bold edges in Fig. 6. As the considered subgraph $K_5^s(K^0)$ is a subdivision of the complete graph on five vertices and $\mathrm{cr}(K_5) = 1$, at least one crossing appears among its edges. Two different subgraphs $K^i$ and $K^j$ do not cross each other and therefore, none of the edges of $K^1$ or $K^2$ belonging to $K_5^s(K^0)$ crosses $K^0$. If an edge of $K^0$ is crossed in $K_5^s(K^0)$ by some other edge of $K_5^s(K^0)$, we have at least one crossing other than two crossings considered in the previous paragraph. Otherwise, as two edges of $K_5$ incident with same vertex do not cross each other and $K^2$ does not cross an edge of $M_K^1$, one of the paths $b_0a_0a_1a_2b_2$ and $e_0f_0f_1f_2e_2$ must cross the edges joining $c_1$ with the vertices of $K^0$. If such a path crosses an edge of $K^1$, then it crosses $K^1$ at least twice and we are done. Otherwise we have at least one crossing between an edge of $M_K^1$ and an edge not belonging to $K^0 \cup M_K^1 \cup K^1 \cup M_K^2 \cup K^2$. The same consideration for the subgraph of $Q_H^1$ consisting of $K^2$ and the paths $c_1c_2$, $c_1b_1b_2$, $c_1d_1d_2$, $c_1e_1e_2$, and $b_2, a_2a_1a_0b_0c_0d_0e_0f_0f_1f_2e_2$ denoted by $K_5^s(K^2)$ confirms that in $D$ there are at least two crossings not considered in the previous paragraph. This proves that in any good drawing of $Q_H^1$ there are at least four crossings and that $\mathrm{cr}(H\Box P_2) = 4$.

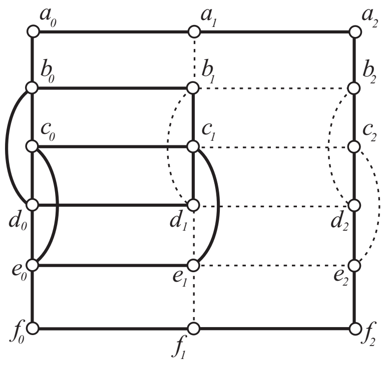


**Fig. 6** The subdivision of $K_5$ in $Q_H^1$.

□

The next lemma help us to prove that the crossing number of the graph $H\Box P_n$ is $2n$ for $n \geq 2$.

**Lemma 4.2.** *If $D$ is a good drawing of the graph $H\Box P_n$, $n \geq 2$, in which every of the subgraphs $H^i$, $i = 0,1,2,...,n$, has at most one crossing on its edges, then there are at least $2n$ crossings in $D$.*

*Proof.* The proof is based on counting the total force of crossings in a drawing of a graph. In a drawing of the graph $H\Box P_n$, let us consider the following types of possible crossings on the edges of $Q_H^i$, $i = 1,2,...,n-1$:

(1) a crossing of an edge in $H^{i-1} \cup M_H^i$ with an edge in $H^{i+1} \cup M_H^{i+1}$,

(2) a crossing of an edge in $M_H^i \cup M_H^{i+1}$ with an edge in $H^i$,

(3) a crossing among the edges of $H^i$.

(4) a crossing among the edges of $H^0 \cup M_H^1$ as well as a crossing of an edge in $H^0$ with an edge in $H^1$,

(5) a crossing among the edges of $H^n \cup M_H^n$ as well as a crossing of an edge in $H^n$ with an edge in $H^{n-1}$.

It is readily seen that every crossing of types (1), (2), and (3) appears in a good drawing of the graph $H\Box P_n$ only on the edges of one subgraph $Q_H^i$. A crossing of type (4) appears only in $Q_H^1$ whereas a crossing of type (5) is a crossing only in $Q_H^{n-1}$. In a good drawing of $H\Box P_n$, we define the force $f(Q_H^i)$ of $Q_H^i$ in the following way: every crossing of type (1), (2), (3), (4), or (5) contributes the value 1 to $f(Q_H^i)$. The total force of the drawing is the sum of $f(Q_H^i)$. It is easy to see that the number of crossings in the drawing is not less than the total force of the drawing. The aim of this proof is to show that if every of the subgraphs $H^i$, $i = 0,1,2,...,n$, has at most one crossing on its edges, then $f(Q_H^i) \geq 2$ for all $i = 2,3,...,n-2$ and $f(Q_H^i) \geq 3$ for $i = 1$ and $i = n-1$.

Consider now the good drawing $D$ of $H\Box P_n$ assumed in Lemma 4.2. This drawing contains the subdrawing of the graph $C_4\Box P_n$ in which none of the 4-cycles separates two other. Otherwise in case when $C_4^i$ separates $C_4^j$ and $C_4^k$, $i < j, i < k$ $(i > j, i > k)$, the cycle $C_4^i$ is crossed by all four edges joining the separated 4-cycles. In the case when $C_4^i$ separates $C_4^j$ and $C_4^k$ for $j < i$ and $k > i$, at least two paths joining $b_j$ with $b_k$ and $e_j$ with $e_k$ cross in $D$ the cycle $C_4^i$. This contradicts the assumption that every of the subgraphs $H^i$ has at most one crossing on its edges. Moreover, two 4-cycles do not cross each other. So, by Lemma 2.1, every subgraph $Q_H^i$ has at least two crossings of types (1), (2), and (3) which are counted in $f(Q_H^i)$ and therefore, $f(Q_H^i) \geq 2$ for all $i = 1,2,\ldots,n-1$.

It remains to prove that $f(Q_H^i) \geq 3$ for $i = 1$ and $i = n-1$. Consider first the subgraph $Q_H^1$. By Lemma 2.1, $\mathrm{cr}_D(C_4^1) + \mathrm{cr}_D(C_4^1, M_C^1 \cup M_C^2) + \mathrm{cr}_D(C_4^0 \cup M_C^1, C_4^2 \cup M_C^2) \geq 2$. Thus, $f(Q_H^1) \geq 2$. If $f(Q_H^1) = 2$, the edges of $H^0 \cup M_H^1$ do not cross each other as well as none of the edges $a_1b_1$ and $e_1f_1$ crosses an edge of $H^0 \cup M_H^1$. A possible crossing in the subdrawing of $C_4^0 \cup M_C^1 \cup C_4^1$ induced by $D$ must appear on the edges of $C_4^1$. But $C_4^1$ can be crossed at most once and as there is no good drawing of the graph $C_4\Box P_1$ with exactly one crossing, $\mathrm{cr}_D(C_4^0 \cup M_C^1 \cup C_4^1) = 0$. The subdrawing $D(C_4^0 \cup M_C^1 \cup C_4^1)$ without crossings is unique with exactly two vertices of $C_4^1$ on the boundary of every region outside $C_4^1$. (This drawing one can obtain from the drawing in Fig. 4(a) by deleting the edges $c_1d_1$ and $c_2d_2$ and by replacing the vertices $b_2, c_2, d_2, e_2$ by the vertices $b_0, c_0, d_0, e_0$.) In $D$, the cycle $C_4^2$ is placed in one region of $D(C_4^0 \cup M_C^1 \cup C_4^1)$ outside $C_4^1$. Moreover, none of the vertices $a_2$ and $f_2$ is placed inside $C_4^1$, otherwise $C_4^1$ is crossed by the edge $a_2b_2$ and by the paths $a_2a_1a_0b_0$ or by the edge $e_2f_2$ and by the path $f_2f_1f_0e_0$, a contradiction. So, at least two edges of $M_C^2$ and at least one of the paths $a_2a_1b_1$ and $f_2f_1e_1$ cross the edges of $D(C_4^0 \cup M_C^1 \cup C_4^1)$. All three crossings contribute the value 1 to $f(Q_H^1)$ and therefore, $f(Q_H^1) \geq 3$. The similar analysis for the subdrawing of $Q_H^{n-1}$ gives $f(Q_H^{i-1}) \geq 3$. So, in $D$ there are at least $3 + \sum_{i=2}^{n-2} f(Q^i) + 3 = 3 + 2(n-3) + 3 = 2n$ crossings. This completes the proof. □

The next theorem determines the crossing number of the graph $H\Box P_n$ for $n \geq 2$.

**Theorem 4.1.** $\mathrm{cr}(H\Box P_n) = 2n$ *for* $n \geq 2$.

*Proof.* The drawing in Fig. 3 shows that $\mathrm{cr}(H\Box P_n) \leq 2n$, because every copy of $H^i$, $i = 1,2,3,...,n-1$, is crossed two times, $H^0$ and $H^n$ are crossed once and there is no other crossing in the drawing. We prove the reverse inequality by induction on $n$. By Lemma 4.1, $\mathrm{cr}(H\Box P_2) = 4$. So, the result is true for $n = 2$. Assume that it is true for $n = k$, $k \geq 2$, and suppose that there is a good drawing of $H\Box P_{k+1}$ with fewer than $2(k+1)$ crossings. By Lemma 4.2, some of the subgraphs $H^i$, $i = 0,1,...,k+1$, must be crossed at least twice. By the removal of all edges of this $H^i$, we obtain a graph homeomorphic to $H\Box P_k$ with fewer than $2k$ crossings or one that contains the subgraph $H\Box P_k$ and has fewer than $2k$ crossings. This contradiction with the induction hypothesis completes the proof. □

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