APPLICATION OF THE CYCLIC PERMUTATION FOR ANALYSIS OF SYNTHESIZED SINUSOIDAL SIGNAL

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SUMMARY

The contribution deals with the analysis of properties of digital sinusoidal signal in both time and frequency domain. Spectral properties of the amplitude quantized or phase-function quantized sinusoidal signal are analyzed in detail. Specific manifestations of the phase-function quantization, that give rise to a restriction in the set of spectral components depending on the number of quantized levels, are highlighted.

The analysis is based on the employment of the index transformation for cyclic permutation and its expression in DFT. This affords the analysis to be confined to just one period of the sinusoidal signal and to get, by using this transformation, the distribution of spectral components for multiple frequencies. Expressions for complex amplitudes of the individual spectral components are derived for amplitude and phase quantization.

Keywords: *quantization, direct digital synthesis, number theory*

1. INDRODUCTION

The contribution presents an analysis of the spectral properties of a quantized sinusoidal signal. This issue arised in connection with the necessity to evaluate quality of sinusoidal signal synthesized in discrete form by the methods of digital synthesis (DFS-DDFS).

There is now a fair amount of literature devoted to this subject (see V.F. Kroupa [1], [2], H.T. Nicholas and H. Samueli [3] and other contributions).

This paper deals only with one particular face of the synthesis, namely the spectral purity of the generated signal. This problem, which is connected with the issues of the frequency stability and phase noise, is a fundamental one.
used to define the optimum architecture based on this DSP core. In this paper we will first present basic blocks of such gateway. Then we will evaluate ST100 capabilities to support expected functions and the corresponding benefits of such integration.

2. FORMULATION OF THE PROBLEM

Ignoring for the moment the questions of technical realization proper of the digital sinusoidal signal generation, two steps can be discerned in the process. First is the creation of a discrete saw function with the repetition frequency given by the steering frequency. Second step consists in the acquisition of the sine function values e.g. by search in the table, with saw function values serving as addresses of the corresponding sine values. Constraints of operation speed and technical realizability impose limits on precision of the phase function and sine values – they must be quantized.

This activity can be conceived as a search for the sine function values in tables that possess a certain finite step in the independent variable values and in which the sine values are entered with a limited number of decimal places. As a consequence, both the phase function and the amplitude of the generated sinusoidal signal are quantized. We will devote only a minor attention to the amplitude values quantization and its influence on the signal's spectral composition – these issues are sufficiently described in the literature. In contrast, the influence of phase function will be studied in detail: it is dominant and exhibits certain specific properties.

The entire treatment is based on the employment of the index transformation for the cyclic permutation which permits the analysis to be limited to one period of the sinusoid with the possibility to extend the results for the multiple frequencies.

2.1. Properties of the cyclically permuted signal

Values of the discrete signal

$$x(n) = \sin(\omega_0 nT) \qquad n = 0,1,2,...,N-1 \qquad \omega_0 = 2\pi / NT \qquad (2\text{-}1)$$

can, in case we consider only each its q^{th} sample, i.e. the signal

$$y(n) = \sin(\omega_0 nqT) \quad n = 0,1,2,...,N-1 \qquad (2\text{-}2)$$

be considered as N samples of a signal with frequency $q\omega_0$.

Due to periodicity of the sine function we can locate these samples into the interval (0,N) using the notation

$$y(n) = \sin((nq - vN)2\pi / N) \qquad (2\text{-}3)$$

Generally, such signal can be written in the form

$$y(n) = x(qn \bmod N) \quad n = 0,1,2,...N-1, \qquad (2\text{-}4)$$

from which it follows that in the interval (0,N) it has, again, N samples arranged, however, in a different order than that for $x(n)$. If one requires that the sequence $y(n)$ should contain the same samples as $x(n)$, the numbers q and N must be mutually prime, i.e. $(q,N)=1$. Such transformation of indices is termed the cyclic permutation with the step q [5] and is written as

$$m = qn \bmod N \qquad (2\text{-}5)$$

Under the condition specified above, to each cyclic permutation there exists an inverse permutation with step r

$$n = rm \bmod N \qquad (2\text{-}6)$$

where the step r fulfills the condition (see Appendix)

$$r = q^{-1} \bmod N. \qquad (2\text{-}7)$$

2.2. DFT and cyclic permutation

If we denote DFT of the original sequence $x(n)$, $n = 0,1,2,\ldots,N$-1

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-jnk2\pi/N} \qquad (2\text{-}8)$$

where $k = 0,1,2,...,N-1$ then, for DFT of the cyclically permuted sequence

$$y(n) = x(qn \bmod N)$$

we get

$$Y(k) = X(rk \bmod N). \qquad (2\text{-}9)$$

This means that to the DFT of the cyclically permuted sequence $x(n)$ with step q, i.e. $y(n)$, corresponds a cyclic permutation of the original sequence $X(k)$ with, however, the step $r = q^{-1} \bmod N$. To the time-domain cyclic permutation corresponds in the frequency domain again a cyclic permutation with an inverse step given by (2-7).

3. DFT SPECTRUM OF THE AMPLITUDE-QUANTIZED SINUSOID

One period of discrete sinusoid quantized in amplitude into H levels can be constructed [6] by the superposition of H partial rectangular signals ${}_hx(n)$, $h = 0,1,2,\ldots,H$-1

$$\begin{aligned} {}_hx(n) &= 0 & n &= 0,1,2,...,\hat{p}_h - 1 \\ &= \Delta & n &= \hat{p}_h, \hat{p}_h + 1,..., N/2 - \hat{p}_h \\ &= 0 & n &= N/2 - \hat{p}_h + 1,..., N/2 + \hat{p}_h - 1 \\ &= -\Delta & n &= N/2 + \hat{p}_h,..., N - \hat{p}_h \\ &= 0 & n &= N - \hat{p}_h + 1,..., N - 1 \end{aligned} \qquad (3\text{-}1)$$

From the definition relation for DFT we than get for $\Delta=1/H$ and the h^{th} level

The value $\hat{p}_h$ is defined as the integer part

$$\hat{p}_h = \text{int}(p_h), \quad p_h = (N/2\pi)\arcsin(h\Delta) \qquad (3\text{-}3)$$

For the DFT of the quantized sinusoid then follows

$$X(k) = \sum_{h=1}^{H-1} {}_hX(k) \quad k = 0,1,2,...N-1 \qquad (3\text{-}4)$$

For DFT signal $y(n)$ created by selecting samples with step q, i.e. the signal $x(qn \bmod N)$, we then get

$$Y(k) = X(rk \bmod N), \quad r = q^{-1} \bmod N \qquad (3\text{-}5)$$

Fig.1 shows the course of $y(n)$ for q=3; Fig.2 presents the module $Y(k)$ and Fig.3 the module $X(k)$.

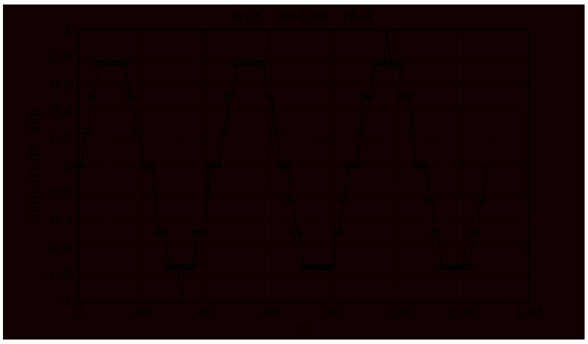

Fig.1 Sinusoidal signal quantized in the amplitude (q=3)

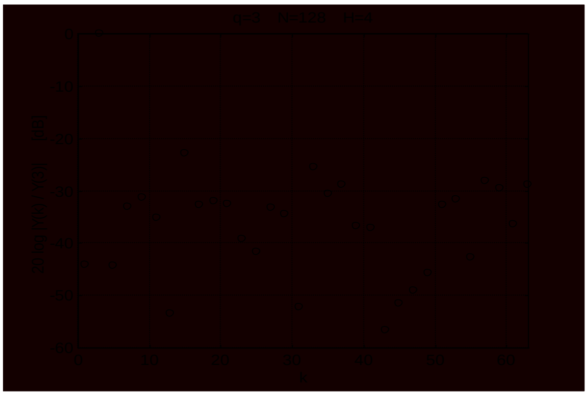

Fig.2 DFT of the sinusoidal signal quantized in the amplitude (q=3)

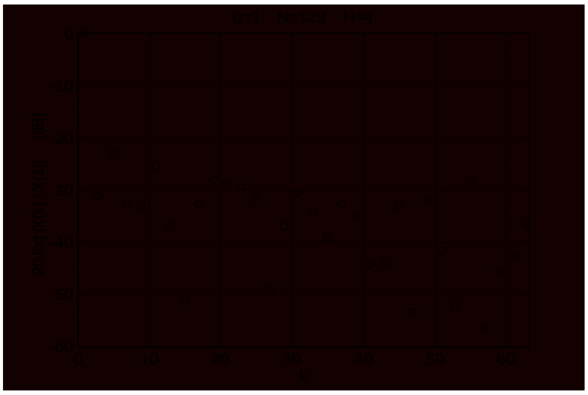

Fig.3 DFT of the sinusoidal signal quantized in the amplitude (q=1)

$${}_hX(k) = \Delta\left(\sum_{n=\hat{p}_h}^{N/2-\hat{p}_h} - \sum_{n=N/2+\hat{p}_h}^{N-\hat{p}_h}\right)e^{-jnk2\pi/N} = \begin{cases} -j2\Delta\cos(k\hat{p}_h 2\pi/N)\cot(k\pi/N) - j2\Delta\sin(k\hat{p}_h 2\pi/N) & \text{for } k = 2v+1 \\ 0 & k = 2v \end{cases} \qquad (3\text{-}2)$$

where $v = 0,1,2,...,N/2-1$

4. DFT SPECTRUM OF SINUSOID QUANTIZED IN THE PHASE

In this case it is the phase function that is quantized; instead of the linear form.

$$\varphi(n) = n\frac{2\pi}{N} \quad n = 0,1,2,...,N-1 \qquad (4\text{-}1)$$

it has the shape of a function with L steps

$$\tilde{\varphi}(n) = \frac{1}{L}\operatorname{int}\left(n\frac{L}{N}\right)2\pi \quad n = 0,1,2,...,N-1. \qquad (4\text{-}2)$$

In case that $N=LM$ with L and M being even numbers, the N samples will be distributed into L intervals each containing M samples of the function $\tilde{\varphi}(n)$. Following expressions hold for the phase function $\tilde{\varphi}(n)$ and the signal $s(n)$ in the particular segments

$$\tilde{\varphi}(n) = \lambda M\,2\pi/N, \quad \lambda M \le n \le (\lambda+1)M-1, \quad \lambda = 0,1,2,...,L-1 \qquad (4\text{-}3)$$

$$s(n) = \sin\tilde{\varphi}(n). \qquad (4\text{-}4)$$

DFT of the signal $s(n)$ can then be written as

$$S(k) = \sum_{\lambda=0}^{L-1}\sum_{n=\lambda M}^{(\lambda+1)M}\sin(\lambda M\,2\pi/N)e^{-jkn2\pi/N} \qquad (4\text{-}5)$$

Following evaluation of the double sum and some further manipulations we get

$$\begin{aligned} S(k) &= -R(k) & k &= vL+1, & v &= 0,1,2,...,M-1 \\ S(k) &= R(k) & k &= vL-1, & v &= 1,2,...,M \\ S(k) &= 0 & k &\ne vL+1, & v &= 0,1,2,...,M-1 \\ & & k &\ne vL-1, & v &= 1,2,...,M \end{aligned}$$

$$R(k) = j\frac{L}{2}\frac{1-e^{-jk2\pi/L}}{1-e^{-jk2\pi/N}} \qquad (4\text{-}6)$$

For amplitudes of the components then follows

$$|S(k)| = \frac{L}{2}\frac{\sin\pi/L}{|\sin k\pi/N|}$$
$$k = vL+1, \quad v = 0,1,2,...,M-1$$
$$k = vL-1, \quad v = 1,2,...,M \qquad (4\text{-}7)$$

The relative amplitude, normalized with respect to the maximum amplitude (k=1) is then

$$\left|\frac{S(k)}{S(1)}\right| = \frac{\sin\pi/N}{|\sin k\pi/N|}, \quad k = vL+1, \quad k = vL-1 \qquad (4\text{-}8)$$

Fig.4 shows the course of $s(n)$ and Fig.5 the corresponding values of DFT $S(k)$.

Given the fact that the component amplitude $S(k)$ decreases for $k < N/2$ with increasing k, one can infer that the closest spectral component to the one with k=1 is the component k=L-1 with the amplitude

$$|S(L-1)| = \frac{L}{2}\frac{\sin\pi/L}{|\sin(L-1)\pi/N|} \qquad (4\text{-}9)$$

The normalized amplitude value for this case is

$$\left|\frac{S(L-1)}{S(1)}\right| = \frac{\sin\pi/N}{|\sin(L-1)\pi/N|}. \qquad (4\text{-}10)$$

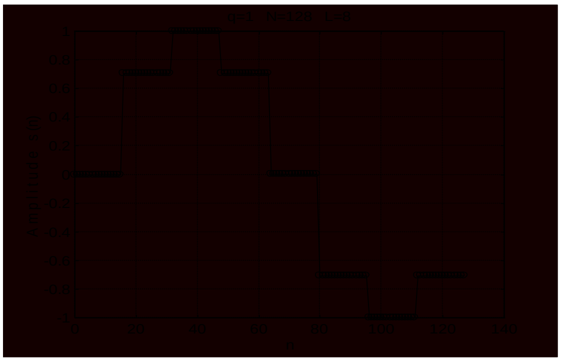

Fig. 4 Sinusoidal signal quantized in the phase (q=1)

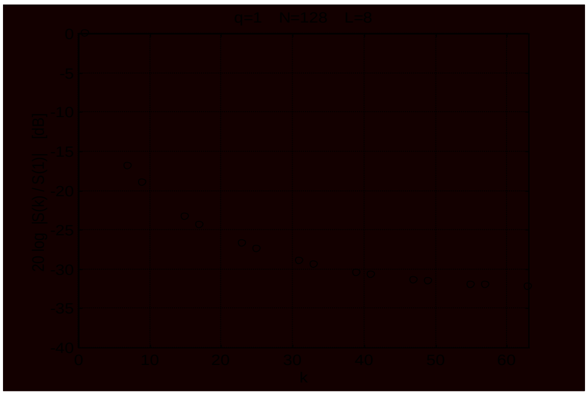

Fig.5 DFT of the sinusoidal signal quantized in the phase (q=1)

Following 2.1, the DFT values of the discrete sinusoid with frequency $q\omega_0$, phase-quantized with N samples, can be expressed with the help of the index transform

Following 2.1, the DFT values of the discrete sinusoid with frequency $q\omega_0$, phase-quantized with N samples, can be expressed with the help of the index transform for cyclic permutation. Denoting values of the sinusoid with frequency $q\omega_0$ as

$$v(n) = s(qn \bmod N), \qquad (4\text{-}11)$$

we get for its DFT

$$V(k) = S(rk \bmod N), \quad r = q^{-1} \bmod N. \qquad (4\text{-}12)$$

Fig.6 demonstrates the effect of index transform $s(n)$ into $v(n)$ and Fig.7 that of the inverse transform $S(k)$ into $V(k)$.

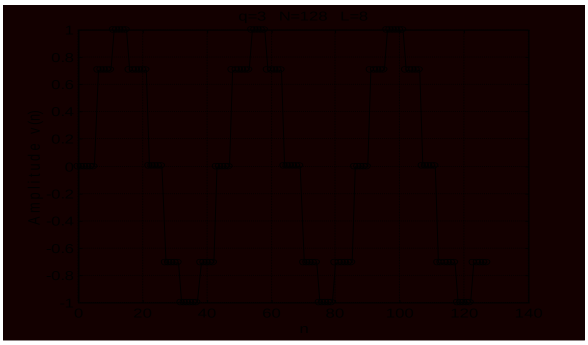

Fig.6 Sinusoidal signal quantized in the phase (q=3)

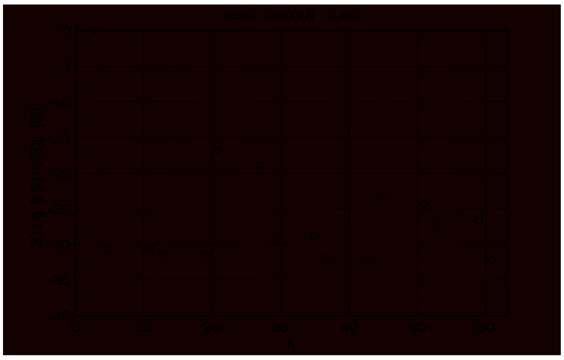

Fig.7 DFT of the sinusoidal signal quantized in the phase (q=3)

5. CONCLUSIONS

Amplitude quantization of the sinusoid was performed in a manner preserving symmetrical properties of the original sinusoid. As a consequence, the sinusoid quantized in this way exhibits, for any number of levels, only the odd spectral components (see(3-2)). The sinusoidal signal with quantized phase function does not possess any symmetrical features, hence its spectrum has a structure that can very widely with the number of quantization levels. It still contains only odd components, but many of them vanish so that the spectrum has a lacunary appearance described by (4-6). It follows from Fig.4 and Fig.6 that quantization of the phase function gives rise to spurious amplitude quantization.

Appendix

To simplify notation, we will employ instead of $n=qm \bmod N$ the form

$$n = \langle qm \rangle_N . \qquad \text{(D-1)}$$

Substituting into this $m = \langle rn \rangle$ instead of m, we get

$$n = \left\langle q \langle rn \rangle_N \right\rangle_N , \qquad \text{(D-2)}$$

from which it follows

$$\langle rq \rangle_N = 1 \qquad \text{i.e.} \qquad r = \left\langle q^{-1} \right\rangle_N \qquad \text{(D-3)}$$

The number r is inverse modulo N to the number q. Solution of the congruence equation (D-2) can be effected in various ways (see, e.g., [7],[4]). We present here the method expressing the inverse value r with the help of the Euler function $\phi(N)$

$$r = \left\langle q^{(\phi(N)-1)} \right\rangle_N \qquad \text{(D-5)}$$

The Euler function of a integer N (which is the number of integers smaller than N, and having no common divisors with N) can be obtained from the canonic expansion (reduction into a product of prime numbers) of N

$$N = \prod_{m=1}^{k} a_m^{\kappa_m} , \quad \phi(N) = N \prod_{m=1}^{k} \left(1 - \frac{1}{a_m} \right). \qquad \text{(D-6)}$$

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BIOGRAPHY

Václav V. Čížek graduated from the Faculty of Electrical Engineering of the Czech Technical University in Prague and took his Ing.(M.Sc.) degree in 1954. In 1961 he obtained from the same University the scientific degree CSc.(corresponding to the Ph.D.). From the 1961 he worked in the Department of network theory in the area of linear system analysis in frequency and time domain. Since 1975 he has been working in the Department of discrete signal processing, especially in the fields of discrete transforms (Fourier, Hilbert etc.), quantization effects in DFT and on applications of the instantaneous frequency concept. His recent interest is in the signal theory and digital frequency synthesis. He is co-author of one textbook and author of two books, one of which has been translated into English. He is also author of more than 70 papers.

Hana Švandová graduated on the Faculty of Electrical Engineering of the Czech Technical University in Prague in 1960 as Ing.(M.Sc.). Since this year she has been employed at the Institute of Radio Engineering and Electronics. During this time she has been concerned with computing and programming in signal processing.