ZERO POINTS OF THE SOLUTIONS OF A DIFFERENTIAL EQUATION

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SUMMARY

Our aim in this paper is to count the zero points of the solutions of the second order differential equation

$$(r(t)u'(t))' + p(t)u(t) = 0. \tag{E^+}$$

The sufficient conditions for the equation (E^+) to be oscillatory are also presented.

Keywords: *canonical and noncanonical equations, comparison theorem*

1. INTRODUCTION

We consider the second order delay differential equation of the form

$$(r(t)u'(t))' + p(t)u(t) = 0. \tag{1}$$

We always assume that r and $p : [t_0, \infty) \to (0, \infty)$ are continuous.

In the sequel we shall restrict our attention to nontrivial solutions of the equations considered. Such a solution is called oscillatory if the set of its zeros is unbounded. Otherwise, it is said to be nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory. On the other hand, we say that (1) is nonoscillatory if all its solutions are nonoscillatory.

We say that equation (1) is in canonical form if

$$\int^{\infty} \frac{\mathrm{d}s}{r(s)} = \infty. \tag{2}$$

On the other hand, if

$$\int^{\infty} \frac{\mathrm{d}s}{r(s)} < \infty \tag{3}$$

then equation (1) is said to be in noncanonical form.

There are two questions we try to solve in the paper. Our interest here is the oscillatory nature of solutions of (1). It si well known that, if one solution of (1) is oscillatory than all solutions share this property. It is thus possible to classify equation (1) as to be oscillatory or nonoscillatory. There are numerous papers devoted to oscillation of (1). See e.g. [5–13].

More precisely than question of whether or not solutions of (1) have infinity zeros in $[T, \infty)$ is the question of how many zeros can a solution have in the prescribed interval $I_1 = [\alpha, \beta] \subset [t_0, \infty]$. Efforts in this direction have been undertaken by several authors, see for example Harris [2], Ohriska [3], Swanson [4]. We make use the following functions in the remainder of this paper:

$$R(t) = \int_t^{t_0} \frac{\mathrm{d}s}{r(t)}, \quad t \ge t_0 \quad \rho(t) = \int_t^{t_0} \frac{\mathrm{d}s}{r(t)}, \quad t \ge t_0,$$

for canonical and noncanonical case of (1), respectively.

2. MAIN RESULTS

The technique we use in the paper is based on the following classical comparison theorem which is due to Sturm [1] and in which equation (1) is compared with the equation

$$(r_1(t)y'(t))' + p_1(t)y(t) = 0. \tag{4}$$

Theorem 2.1. *Assume that*

$$r(t) \le r_1(t), \quad t \in I_1, \tag{5}$$

$$p(t) \ge p_1(t), \quad t \in I_1. \tag{6}$$

If y is a nontrivial solution of (4) *with the property $y(t_1) = y_1(t) = 0$, $t_1 < t_2$, where $t_1, t_2 \in I_1$ are the couple of its adjoining zeros, then every solution u of* (1) *has at least one zero in* (t_1, t_2) *or equations* (1) *and* (4) *are equivalent in* $[t_1, t_2]$ *and y and u are linearly depend.*

Corollary 2.1. *Assume that* (5) *and* (6) *hold. Let u and y be arbitrary nontrivial solutions of* (1) *and* (4)*, respectively.*

(i) If y has at least m zero points in I_1 then u has at least $m-1$ zero points in I_1.

(ii) If u has at most k zero points in I_1 then y has at most $k+1$ zero points in I_1.

Now we are prepared to provide the lower estimate of zero points of any solution of (1).

Theorem 2.2. *Assume that there exist a real $a > \frac{1}{4}$ such that*

$$R^2(t)r(t)p(t) \ge a, \quad for \quad t \in [\alpha, \beta] \quad and \quad \alpha > t_0.$$

Then the number of the zero points of each solution of (1) *on the interval* $[\alpha, \beta]$ *equals at least*

$$\left\lfloor \frac{\sqrt{4a-1}}{2\pi} \ln \frac{R(\beta)}{R(\alpha)} \right\rfloor.$$

Proof. Let us consider the following auxiliary equation

$$(r(t)y'(t))' + \frac{a}{r(t)R^2(t)}y(t) = 0, \quad t \in I_1. \tag{7}$$

It is easy to see that the couple of the functions

$$y_1(t) = \sqrt{R(t)}\cos\left(\frac{\sqrt{4a-1}}{2}\ln R(t)\right),$$
$$y_2(t) = \sqrt{R(t)}\sin\left(\frac{\sqrt{4a-1}}{2}\ln R(t)\right)$$

are linearly independent solutions of (7). Consider such nontrivial solution $y_3(t)$ of (7) satisfying $y_3(\alpha) = 0$. Then there are constants c_1 and c_2 such that $y_3(t) = c_1y_1(t) + c_2y_2(t)$. Taking into account properties of $\sin\left(\frac{\sqrt{4a-1}}{2}\ln R(t)\right)$ and $\cos\left(\frac{\sqrt{4a-1}}{2}\ln R(t)\right)$ it is easy to see that $y_3(t)$ has $\left\lfloor \frac{\sqrt{4a-1}}{2\pi}\ln\frac{R(\beta)}{R(\alpha)}\right\rfloor + 1$ zero points on the interval $[\alpha, \beta]$. Noting that $p(t) \geq \frac{a}{r(t)R^2(t)}$ for $t \in I_1$ and applying Corollary 2.1 (i) to (1) and (7) we immediately get our assertion.

□

Remark 2.1. *Theorem 1 is a complement of the Ohriska's result [2, Theorem 2.1] providing an estimate of the zero points of the solutions of* (1) *under more stronger condition imposed on the function p.*

Having the lower estimate of zero points of solutions of equation (1) on the prescribed interval, we can immediately get oscillatory criterion for (1).

Corollary 2.2. *Let* $\beta = \infty$ *and* (1) *be in canonical form. Further suppose that all assumptions of Theorem* (1) *are satisfied. Then equation* (1) *is oscillatory.*

Remark 2.2. *Note that if the condition* (2) *is not satisfied we cannot use the estimate of the zero points of the solutions of* (1) *presented in Theorem 1 to deduce oscillatory nature of equation* (1).

We provide another lower estimate of zero points of solutions of (1).

Theorem 2.3. *Assume that*

$$r(t)p(t) \geq a \geq 0, \quad for \quad t \in [\alpha, \beta].$$

Then the number of the zero points of each solution of (1) *on the interval* $[\alpha, \beta]$ *equals at least*

$$\left\lfloor \frac{\sqrt{a}}{\pi}\,(R(\beta) - R(\alpha))\right\rfloor.$$

Proof. Note that the couple of the functions

$$y_1(t) = \cos\left(\sqrt{a}R(t)\right),$$
$$y_2(t) = \sin\left(\sqrt{a}R(t)\right),$$

are solutions of the following auxiliary equation

$$(r(t)y'(t))' + \frac{a}{r(t)R^2(t)}y(t) = 0, \quad t \in I_1.$$

Next we can follow all steps of the proof of Theorem 2.2 so details are left for reader.

□

In the following result we deduce the upper estimate of zero points of any solution of (1).

Theorem 2.4. *Assume that there exist a real* $b > \frac{1}{4}$ *such that*

$$R^2(t)r(t)p(t) \leq b, \quad for \quad t \in [\alpha, \beta].$$

Then the number of the zero points of each solution of (1) *on the interval* $[\alpha, \beta]$ *equals at most*

$$\left\lfloor \frac{\sqrt{4b-1}}{2\pi}\ln\frac{R(\beta)}{R(\alpha)}\right\rfloor + 1.$$

Proof. Note that the functions

$$u_1(t) = \sqrt{R(t)}\cos\left(\frac{\sqrt{4b-1}}{2}\ln R(t)\right),$$
$$u_2(t) = \sqrt{R(t)}\sin\left(\frac{\sqrt{4b-1}}{2}\ln R(t)\right)$$

are linearly independent solutions of the equation

$$(r(t)u'(t))' + \frac{b}{r(t)R^2(t)}u(t) = 0, \quad t \in [\alpha, \beta]. \tag{8}$$

We shall proceed similarly as in the proof of Theorem 1. We consider a solution $u_3(t)$ of (8) with property $u_3(\alpha) = 1$. Again $u_3(t) = c_1u_1(t) + c_2u_2(t)$, employing properties of $\sin\left(\frac{\sqrt{4a-1}}{2}\ln R(t)\right)$ and $\cos\left(\frac{\sqrt{4a-1}}{2}\ln R(t)\right)$ we are sure that $u_3(t)$ has $\left\lfloor \frac{\sqrt{4a-1}}{2\pi}\ln\frac{R(\beta)}{R(\alpha)}\right\rfloor$ zero points on the interval $[\alpha, \beta]$. Since $p(t) \leq \frac{b}{r(t)R^2(t)}$ for $t \in [\alpha, \beta]$ our assertion follows from Corollary 2.1 (ii) applied to to (8) and (1).

□

As a simple consequence of the previous result we get the following nonoscillatory criterion for (1).

Corollary 2.3. *Let* $\beta = \infty$ *and* (3) *hold. Further suppose that all assumptions of Theorem* (2.4) *are satisfied. Then equation* (1) *is nonoscillatory.*

Corollary 2.4. *Let* (2) *hold. Assume that*

$$R^2(t)r(t)p(t) \leq \frac{1}{4} \quad for \quad t \in [\alpha, \infty).$$

Then equation (1) *is nonoscillatory.*

Proof. It is easy to see that $u_1(t) = \sqrt{R(t)}$ is a nonoscillatory solution of the auxiliary equation

$$(r(t)u'(t))' + \frac{1}{4r(t)R^2(t)}u(t) = 0, \quad t \in [\alpha, \infty). \tag{9}$$

Taking into account that $p(t) \leq \frac{1}{4r(t)R^2(t)}$ for $t \in [\alpha, \infty)$ and applying Corollary 2.1 (ii) to (9) and (1) we see that any solution of (1) has at most 1 zero point in $[\alpha, \infty)$ that is it is nonoscillatory.

□

Now we turn our attention to lower estimate of zero points of any solution of noncanonical case of (1).

Theorem 2.5. *Let* (3) *hold. Assume that there exist a real* $a > \frac{1}{4}$ *such that*

$$\rho^2(t)r(t)p(t) \geq a, \quad \text{for } t \in [\alpha,\beta].$$

Then the number of the zero points of each solution of (1) *on the interval* $[\alpha,\beta]$ *equals at least*

$$\left\lfloor \frac{\sqrt{4a-1}}{2\pi} \ln \frac{\rho(\alpha)}{\rho(\beta)} \right\rfloor.$$

Proof. Let us consider the equation

$$(r(t)y'(t))' + \frac{a}{r(t)\rho^2(t)} y(t) = 0, \quad t \in I_1. \tag{10}$$

One can see that the functions

$$y_1(t) = \sqrt{\rho(t)}\cos\left(\frac{\sqrt{4a-1}}{2}\ln\frac{1}{\rho(t)}\right),$$
$$y_2(t) = \sqrt{\rho(t)}\sin\left(\frac{\sqrt{4a-1}}{2}\ln\frac{1}{\rho(t)}\right)$$

are linearly independent solutions of (10). Consider such nontrivial solution $y_3(t)$ of (10) satisfying $y_3(\alpha) = 0$. Then $y_3(t)$ has $\left\lfloor \frac{\sqrt{4a-1}}{2\pi} \ln \frac{\rho(\alpha)}{\rho(\beta)} \right\rfloor + 1$ zero points on I_1. Since $p(t) \geq \frac{a}{r(t)\rho^2(t)}$ for $t \in I_1$, Corollary 2.1 (i) implies our assertion.

□

Note that the estimate of the zero points of the solutions of (1) presented in Theorem (2.6) can be used to deduce the oscillation of the equation (1) as the following corollary shows:

Corollary 2.5. *Let* $\beta = \infty$ *and* (1) *be in noncanonical form. Further suppose that all assumptions of Theorem* (4) *are satisfied. Then equation* (1) *is oscillatory.*

We complete our considerations with providing the upper estimate of zero points of any solution of noncanonical (1).

Theorem 2.6. *Let* (3) *hold. Assume that there exist a real* $b > \frac{1}{4}$ *such that*

$$\rho^2(t)r(t)p(t) \leq b, \quad \text{for} \quad t \in [\alpha,\beta].$$

Then the number of the zero points of each solution of (1) *on the interval* $[\alpha,\beta]$ *equals at most*

$$\left\lfloor \frac{\sqrt{4b-1}}{2\pi} \ln \frac{\rho(\alpha)}{\rho(\beta)} \right\rfloor + 1.$$

Proof. Since the functions

$$u_1(t) = \sqrt{\rho(t)}\cos\left(\frac{\sqrt{4b-1}}{2}\ln\frac{1}{\rho(t)}\right) \quad \text{and}$$
$$u_2(t) = \sqrt{\rho(t)}\sin\left(\frac{\sqrt{4b-1}}{2}\ln\frac{1}{\rho(t)}\right)$$

are linearly independent solutions of the equation

$$(r(t)u'(t))' + \frac{b}{r(t)\rho^2(t)} u(t) = 0, \quad t \in [\alpha,\beta].$$

and $p(t) \leq \frac{b}{r(t)\rho^2(t)}$ for $t \in I_1$, the conclusion of the theorem follows from Corollary 2.1.

□

Corollary 2.6. *Let* (3) *hold. Assume that*

$$\rho^2(t)r(t)p(t) \leq \frac{1}{4} \quad \text{for } t \in [\alpha,\infty).$$

Then equation (1) *is nonoscillatory.*

In the following example applying Theorems 2.2 and 2.6 we provide upper and lower estimate for zero points of solutions of considered equation.

Example 2.1. *Let us consider the second order differential equation*

$$(tu')' + \frac{5\ln t}{2t + 2t\ln^3 t} u(t) = 0, \quad t \geq 1. \tag{11}$$

It is easy to see that

$$R^2(t)r(t)p(t) = \frac{5\ln^3 t}{2 + 2\ln^3 t} \leq \frac{5}{2} \quad \text{for} \quad t \geq 1$$

and

$$R^2(t)r(t)p(t) \geq \frac{5}{4} \quad \text{for} \quad t \geq e.$$

Then the number of the zero points of each solution of (11) *on the interval* $[e,e^k]$, $k > 1$ *equals by the Theorem 2.2 at least*

$$\left\lfloor \frac{\ln k}{\pi} \right\rfloor$$

and by the Theorem 2.4 at most

$$1 + \left\lfloor \frac{3\ln k}{2\pi} \right\rfloor.$$

Our results here can be compared with a result of Harris [1, Crollary 8] in which an estimate od the zero points of the solutions of (1) is provided but the condition $rp \in C^2([\alpha,\beta])$ is required.

Remark 2.3. *If we are interested only in counting the zero-points of the solutions of* (1)*, we can use as comparative equation any equation whose solutions can be found explicitly. For details see next two examples.*

Example 2.2. *We consider the second order auxiliary differential equation*

$$(r(t)y'(t))' + \frac{1}{r(t)}\left(e^{2R(t)} - \frac{1}{4}\right) y(t) = 0, \tag{12}$$

where $t \in [\alpha,\beta]$. *We can directly verify that*

$$y_1(t) = \frac{1}{\sqrt{e^{R(t)}}} \sin e^{R(t)} \quad \text{and}$$
$$y_2(t) = \frac{1}{\sqrt{e^{R(t)}}} \cos e^{R(t)}$$

are the couple of the linearly independent solutions of (12). *Thus Eq.* (12) *has a solution with* $\left\lfloor \frac{e^{R(\beta)} - e^{R(\alpha)}}{\pi} \right\rfloor + 1$ *zero points. Therefore, if* (2) *holds and* $r(t)p(t) \geq e^{2R(t)} - \frac{1}{4}$ *then by Corollary 2.1 the number of the zero points of each solution of* (1) *on the interval* $[\alpha,\beta]$ *equals at least*

$$\left\lfloor \frac{e^{R(\beta)} - e^{R(\alpha)}}{\pi} \right\rfloor.$$

Example 2.3. *Consider the second order auxiliary differential equation*

$$(r(t)y'(t))' + \frac{e^{\frac{2}{\rho(t)}} - \frac{1}{4}}{\rho^4(t)r(t)} y(t) = 0, \quad t \in [\alpha, \beta]. \qquad (13)$$

If (3) *holds then*

$$y_1(t) \;=\; e^{\frac{-1}{2\rho(t)}} \rho(t) \sin e^{\frac{1}{\rho(t)}} \quad \text{and}$$
$$y_2(t) \;=\; e^{\frac{-1}{2\rho(t)}} \rho(t) \cos e^{\frac{1}{\rho(t)}}$$

are couple of the linearly independent solutions of (13). *And so, if* $\rho^4(t)r(t)p(t) \geq e^{\frac{2}{\rho(t)}} - \frac{1}{4}$ *then by Corollary 2.1 the number of the zero points of each solutions of* (1) *on the interval* $[\alpha, \beta]$ *equals at least*

$$\left\lfloor \frac{e^{\frac{1}{\rho(\beta)}} - e^{\frac{1}{\rho(\alpha)}}}{\pi} \right\rfloor .$$

3. CONCLUSION

In this paper we have provided upper and lower estimates of zero points of all nontrivial solutions of the second order linear equations in both canonical and noncanonical cases. As a consequences of these estimates we get oscillation and nonoscillation criteria for equations considered. More over using theory presented here we can easily get estimates of zero points of (1) provided that we can solve explicitely various auxiliary equations of the form (1).

We leave an open problem how to extend our results here to even order higher differential equations and to provide estimates of zero points of their solutions.

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