FAST ALGORITHM FOR EXTREMAL BIPARAMETRIC EIGENPROBLEM

*Martin GAVALEC, **Ján PLAVKA
*Department of Information Technologies, Faculty of Informatics and Management, University of Hradec Králové, Rokitanského 62, 500 03 Hradec Králové, Czech Republic E-mail: Martin.Gavalec@tuke.sk
**Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, B. Němcovej 32, 042 00 Košice, tel. 095/602 3256, E-mail: Jan.Plavka@tuke.sk

SUMMARY

Denote $a \oplus b = \max(a,b)$, and $a \otimes b = a + b$ for $a, b \in R$ and extend this pair of operations to matrices and vectors in the same way as in conventional linear algebra, that is if $A = (a_{ij}), B = (b_{ij}), C = (c_{ij})$ are real matrices or vectors of compatible sizes then $C = A \otimes B$ if $c_{ij} = \Sigma_k^{\oplus} a_{ik} \otimes b_{kj}$ for all i, j. For arbitrary parameters α, β and given square matrices $A = (a_{ij})$, we study the Biparametric Eigenproblem, i.e. problem of finding all $x_{\alpha,\beta} = (x_1(\alpha,\beta), x_2(\alpha,\beta), \ldots, x_n(\alpha,\beta))$ and $\lambda_{\alpha,\beta}$, satisfying

$$A(\alpha,\beta) \otimes x_{\alpha,\beta} = \lambda_{\alpha,\beta} \otimes x_{\alpha,\beta}$$

where $A(\alpha,\beta) = (a_{ij}(\alpha,\beta))$, $a_{ij}(\alpha,\beta) = a_{ij} + \alpha$ for $j = 1$, $a_{ij}(\alpha,\beta) = a_{ij} + \beta$ for $j = 2$ and $a_{ij}(\alpha,\beta) = a_{ij}$ otherwise. We introduce some properties of general Biparametric Eigenproblem and an $O(n^5)$ algorithm which gives solutions of it.

Keywords: *extremal eigenvalue, eigenvector*

1. INTRODUCTION

Let G=$(G, \otimes, \leq)$ be a linearly ordered, commutative group with neutral element $e = 0$. We suppose that G is radicable, i.e. for every integer $t \geq 1$ and for every $a \in G$, there exists a (unique) element $b \in G$ such that $b^t = a$. We denote $b = a^{1/t}$.
Throughout the paper $n \geq 1, m \geq 1$ are given integers. The set of $n \times m$ matrices over G is denoted by $G(n,m)$. We introduce further a binary operation $\oplus$ on G by the formula

$$a \oplus b = \max(a,b) \quad \text{for all } a, b \in G.$$

The triple $(G, \oplus, \otimes)$ is called *max-algebra*. If G=$(G, \otimes, \leq)$ is additive group of real numbers, then $(G, \oplus, \otimes)$ is called *max-plus algebra* (often used in applications). The operations $\oplus, \otimes$ are extended to the matrix-vector algebra over G by the direct analogy to the conventional linear algebra. We extend G by a new element $-\infty$, we denote $G \cup \{-\infty\}$ by $\bar{G}$ and extend $\otimes$ and $\leq$ to $\bar{G}$: $a \otimes -\infty = -\infty \otimes a = -\infty$ and $-\infty < a$ for all $a \in G$. The symbol $diag(d_1, d_2, \ldots, d_n)$ denotes the matrix D with diagonal elements equal to $d_1, d_2, \ldots, d_n$ and off-diagonal elements equal to $-\infty$. This matrix D will be called *diagonal* if all $d_1, d_2, \ldots, d_n \in G$. If $D = diag(\alpha, \beta, e, \ldots, e)$, α, $\beta \in G$ and $A \in G(n,n)$ denote $A(\alpha,\beta) = A \otimes D$.
The aim of this paper is to present a description of the eigenvalues and to analyze the eigenspace with respect to α, β. Below, we summarize and recall some of the main results. First we introduce the necessary notation.
Let N=$\{1, 2, \ldots, n\}$ and let C_n be the set of all cyclic permutations defined on nonempty subsets of N. For a cyclic permutation $\sigma = (i_1, i_2, \ldots, i_l) \in C_n$ and for $A \in G(n,n)$, we denote l, the length of σ by $l(\sigma)$ and define

$$w_A(\sigma) = a_{i_1 i_2} \otimes a_{i_2 i_3} \otimes \cdots \otimes a_{i_l i_1}, \;\; \mu_A(\sigma) = w_A(\sigma)^{1/l(\sigma)},$$

$$\lambda(A) = \sum_{\sigma \in C_n}^{\oplus} \mu_A(\sigma)$$

where $\sum^{\oplus}$ denotes the iterated use of the operation $\oplus$. The *eigenproblem* in max-algebra is formulated as follows: Given $A \in G(n,n)$, find $x \in G(n,1)$ and $\lambda(A) \in G$ satisfying

$$A \otimes x = \lambda(A) \otimes x.$$

This problem was treated by several authors during the sixties, c.g. [3,6], survey of the results concerning this and similar eigenproblems can be found in [16, 17].
The $\ell -$ *parametric eigenproblem* in max-algebra was studied in [15] and is defined similarly as the eigenproblem but entries in the first ℓ columns depend on the same parameter.
The *biparametric eigenproblem* in max-algebra is defined as follows: For two arbitrary parameters $\alpha, \beta \in G$ and given $A \in G(n,n)$ find $x_{\alpha,\beta} \in G(n,1)$ and $\lambda(A(\alpha,\beta)) \in G$ satisfying

$$A(\alpha,\beta) \otimes x_{\alpha,\beta} = \lambda(A(\alpha,\beta)) \otimes x_{\alpha,\beta}.$$

The symbol $D_A = (V, E)$ stands for a complete, arc-weighted digraph associated with A. The node set of D_A is N, and the weight of any arc (i, j) is a_{ij}. Throughout the paper, by a cycle in the digraph we mean an elementary cycle or a loop, and by path we mean a nontrivial elementary path, i.e. an elementary path containing at least one arc. Evidently, we will use the same notation, as well as the concept of weight, both for cycles and cyclic permutations. A cycle $\sigma \in C_n$ is *optimal*, if $\mu_A(\sigma) = \lambda(A)$, a node in D_A is called an *eigennode* if it is contained in at least one optimal cycle; E_A stands for the set of all eigennodes in D_A.

Theorem 1.1. [4] *Each square matrix has at most one eigenvalue. If G is radicable then every square matrix A has exactly one eigenvalue (denoted as $\lambda(A)$*

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in what follows). This unique eigenvalue is equal to the maximal average weight of cycles in D_A.

Theorem 1.2. [4] *Let G be radicable, $A \in G(n,n)$ and $\alpha \in G$. Then*

$$\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A).$$

The problem of finding the eigenvalue $\lambda(A)$ is also called the *maximum cycle mean problem* and it has been studied by several authors [1–6, 8, 10, 12–15]. Various algorithms for solving this problem are known, that of Karp [10] having the best worst-case performance $O(n^3)$ and Howard's algorithm [9] of unproved computational complexity showing excellent algorithmic performance. For $B \in G(n,n)$ we denote by $\Delta(B)$ the matrix $B \oplus B^2 \oplus \ldots \oplus B^n$ where B^s stands for the s-fold iterated product $B \otimes B \otimes \ldots \otimes B$.
Let $A_\lambda = \lambda(A)^{-1} \otimes A$. (The upper index -1 denotes the inverse element of $\lambda(A)$ in the sense of the group operation $\otimes$). It is shown in [4] that the matrix $\Delta(A_\lambda)$ contains at least one column, the diagonal element of which is e. Every such column is an eigenvector of the matrix A, it is called a *fundamental eigenvector* of the matrix A. The set of all fundamental eigenvectors will be denoted by F_A and its cardinality is denoted by $q = |F_A|$. We say that $x, y \in F_A$ are equivalent if $x = \alpha \otimes y$ for some $\alpha \in G$. In what follows $s(A)$ denotes the set of all eigenvectors of A, so called *eigenspace* of A.

Theorem 1.3. [4] *Let $A \in G(n,n)$. Then*

$$s(A) = \Big\{ \sum_{i=1}^{q} \alpha_i \otimes g_i;\ \alpha_i \in G,\ g_i \in F_A,\ i = 1,2,\ldots,q \Big\}.$$

It follows from the definition of equivalent fundamental eigenvectors that the set F_A in *Theorem 1.3* can be replaced by any maximal set F'_A of fundamental eigenvectors such that no two of them are equivalent. Every such set F'_A will be called a complete set of generators (of the eigenspace).

Theorem 1.4. [4] *Let $g_1, g_2, \ldots, g_n$ denote the columns of the matrix $\Delta(A_\lambda)$. Then*

(i) $j \in E_A$ if and only if $g_j \in F_A$

(ii) g_i, g_j are equivalent members of F_A if and only if the eigennodes i, j are contained in a common optimal cycle.

Let be $\Delta(A_\lambda) = (\xi_{ij})$. It follows from the definition of $\Delta(A_\lambda)$ that ξ_{ij} is the weight of the heaviest path from i to j in D_A. Hence, $\Delta(A_\lambda)$ can be computed in $O(n^3)$ operations using the Floyd-Warshall algorithm [11]. By trivial search and comparisons one can then find a complete set of fundamental eigenvectors among the columns of $\Delta(A_\lambda)$, using at most $O(n^3)$ operations.
The next assertion follows straightforwardly from the definition of $\Delta(A_\lambda)$.

Theorem 1.5. *Let $d \in G$, $A \in G(n,n)$ and $D = diag\{d, \ldots, d\}$. Then*

$$\Delta(A_\lambda) = \Delta((A \otimes D)_\lambda).$$

2. BIPARAMETRIC MAXIMAL CYCLE MEAN PROBLEM

The aim of this section is to investigate the above *biparametric maximum cycle mean* for $A(\alpha,\beta)$, where A is given matrix and $\alpha,\ \beta \in G$. W.o.l.g. we will deal with case $G = R$ and with a given matrix A having $\lambda(A) = 0$ (*Theorem 1.2*). Suppose that a given matrix A has the following block diagonal form

$$A = \begin{pmatrix} B & . \\ . & C \end{pmatrix}$$

where B and C are 2×2 and $(n-2 \times (n-2)$ square submatrices of A, respectively. The next theorem describes very easy provable property and the bound of $\lambda(A(\alpha,\ \beta))$.

Theorem 2.1. *If $\alpha,\ \beta \geq 0$ then* $\lambda(A(\alpha,\beta)) \geq \max(\lambda(B(\alpha,\beta)),\ \lambda(C))$.

For a given matrix $A = (a_{kl}) \in G(n,n)$, $i \in N$, a cyclic permutation $\sigma = (i_1, \ldots, i_s)$, $|\{i_1, i_2, \ldots, i_s\} \bigcap \{1,2,\ldots,\ell\}| = \{1,\ldots,k\}$ denote by

$$m_s^{1,\ldots,k} = \max_{\sigma \in C_n^{1,\ldots,k}} \mu_A(\sigma) =$$

$$= \max_{\sigma \in C_n^{1,\ldots,k}} \left\{ \frac{a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_s i_1}}{s} \right\},$$

where $C_n^{1,\ldots,k} \subset C_n$ is the set of all cyclic permutations defined on subsets of N containing just elements $1, \ldots, k$. Denote the following functions by

$$m_s^{\{1\}}(\alpha) = m_s^1 + \frac{\alpha}{s}, \quad m_s^{\{2\}}(\beta) = m_s^2 + \frac{\beta}{s},$$

$$m_s^{\{1,2\}}(\alpha,\beta) = m_s^{1,2} + \frac{\alpha+\beta}{s},$$

and the sets by

$$P_{\geq}^{\{j\}}(v) = \{(\alpha,\beta) \in R \times R;\ m_v^{\{j\}}(\alpha,\beta) \geq$$

$$\max_{s \in N} \{m_s^{\{1,2\}}(\alpha,\beta);\ m_s^{\{1\}}(\alpha);\ m_s^{\{2\}}(\beta);\ \lambda(C)\}$$

$$P_{>}^{\{j\}}(v) = \{(\alpha,\beta) \in R \times R;\ m_v^{\{j\}}(\alpha,\beta) >$$

$$\max_{s \in N,\ s \neq v} \{m_s^{\{1,2\}}(\alpha,\beta);\ m_s^{\{1\}}(\alpha);\ m_s^{\{2\}}(\beta);\ \lambda(C)\}$$

for $j \in \{\{1\},\{2\},\{1,2\}\}$.

Theorem 2.2. *Let $\alpha,\ \beta \in P_{\geq}^{j}(v)$ for $j \in \{\{1\},\{2\},\{1,2\}\}$. Then $\lambda(A(\alpha,\beta)) = m_v^j(\alpha,\beta)$.*

Proof. Suppose $\alpha,\ \beta \in P_{\geq}^{j}(v)$ for $j \in \{\{1\},\{2\},\{1,2\}\}$. Since the set C_n of all cyclic permutations is possible to split into four disjoint classes as follows:

$$C_n = C_n^1 \cup C_n^2 \cup C_n^{1,2} \cup C_n'^{1,2},$$

where C_n^1, C_n^2, $C_n^{1,2}$ are the sets of all cyclic permutations defined on subsets of N containing just elements from the set $\{1,2\}$. The set $C_n^{'1,2}$ is the set of all cyclic permutations defined on subsets of N not containing element 1 and 2. Then according to the definition of $P_{\geq}^{\{j\}}$ we get:
$m_v^j(\alpha,\beta) \geq \max_{s\in N}\{m_s^{\{1,2\}}(\alpha,\beta);\ m_s^{\{1\}}(\alpha);\ m_s^{\{2\}}(\beta);\ \lambda(C)\} = \max_{\sigma\in C_n}\mu_{A(\alpha,\beta)}(\sigma)$.
□

3. COMPUTATIONAL ASPECT

To solve effectively the biparametric maximum cycle mean problem is necessary to have efficient algorithm for computing $m_s^{1,2}$, m_s^1, m_s^2 whereby the values m_s^1, m_s^2 is possible to compute by using the matrix $W = (w_{i,j}^u)$ with elements which describes the weight of the heaviest pathes from node i to the node $j \in \{1,2\}$ of length u in D_A. Denote by $B_A = (b_{ij})$ and $C_A = (c_{ij})$ the $n\times n$ matrix which arose from the matrix A by replacing all entries of first row and first column and second row and second column by $-\infty$, respectively.

If $b_j^1 = (b_{1j}^1,\dots,b_{nj}^1)$ is $j-th$ column of B_A and $c_j^1 = (c_{1j}^1,\dots,c_{nj}^1)$ is $j-th$ column of C_A then define the sequence of vectors as follows:

$$b_j^{k+1} = B_A \otimes b_j^k,\ \ c_j^{k+1} = C_A \otimes c_j^k,$$

for $k = 1,\dots,n-1$ and $m_s^1 = \frac{c_{jj}^s}{s}$, $m_s^2 = \frac{b_{jj}^s}{s}$.
To compute $m_s^{1,2}$ is harder problem. For this we will use the following theorem.

Theorem 3.1. $m_2^{1,2} = \frac{a_{12}+a_{21}}{2}$,
$m_3^{1,2} = \max_{k\geq 3}(\frac{a_{1k}+a_{k2}+a_{21}}{3}, \frac{a_{12}+a_{2k}+a_{k1}}{3})$ *and*
$m_s^{1,2} = \max_{u,v,s=k+l+2}\frac{c_{1u}^k+a_{u2}+b_{2v}^l+a_{v1}}{k+l+2}$ *for* $s\geq 4$.

Proof. Suppose that

$$\sigma = (1,i_2,\dots,i_v,2,i_{v+2},\dots,i_s),\ l(\sigma) = s$$

and $\mu(\sigma) > m_s^{1,2} = \frac{c_{1u}^k+a_{u2}+b_{2v}^l+a_{v1}}{k+l+2}$. Then the inequality $a_{1i_1}+a_{i_1i_2}+\dots+a_{i_v2}+a_{2i_{v+2}}+\dots+a_{i_s1} > c_{1u}^k + a_{u2}+b_{2v}^l+a_{v1}$ implies either $a_{1i_1}+a_{i_1i_2}+\dots+a_{i_v2} > c_{1u}^k+a_{u2}$ or $a_{2i_{v+2}}+\dots+a_{i_s1} > b_{2v}^l+a_{v1}$ what is a contradiction with the definition of $m_s^{1,2}$.
□

The best worst-case performance for the computing m_s^1, m_s^2 and $m_s^{1,2}$ is $O(n^3)$.

4. BIPARAMETRIC EIGENVECTORS

To compute the eigenvectors of biparametric matrix we use the following very easy proving theorem.

Theorem 4.1. *Let* $\alpha,\ \beta \in P_{>}^j(v)$ *for* $j \in \{\{1\},\{2\},\{1,2\}\}$. *Then* $|F_A'(\alpha,\beta)| = 1$.

If $|F_A'(\alpha,\beta)| = 1$ then we shall analyze three possibilities.
1. Let $\lambda(A(\alpha,\beta)) = m_s^1 + \frac{\alpha}{s}$ then

$$\xi_{j1}(\alpha,\beta) = \max_k\left\{m_{j1}^k - k(m_s^1+\frac{\alpha}{s})\right\}$$

where m_{j1}^k is the maximal weight of the path from node j to the node 1 over k edges in corresponding D_{A_λ}.
2. Let $\lambda(A(\alpha,\beta)) = m_s^2 + \frac{\beta}{s}$ then

$$\xi_{j2}(\alpha,\beta) = \max_k\left\{m_{j2}^k - k(m_s^2+\frac{\beta}{s})\right\}$$

where m_{j2}^k is the maximal weight of the path from node j to the node 2 over k edges in corresponding D_{A_λ}.
3. Let $\lambda(A(\alpha,\beta)) = m_s^{1,2} + \frac{\alpha+\beta}{s}$ then

$$\xi_{j1}(\alpha,\beta) = \max_k\left\{M_{j1}^k - k(m_s^{1,2}+\frac{\alpha+\beta}{s})\right\}$$

where M_{j1}^k is the maximal weight of the path from node j to the node 1 over k edges in corresponding D_{A_λ}. Since we have entries from the computing process of eigenvalue we can formulate the crucial assertion of this section.

Theorem 4.2. *1. Let* $\lambda(A(\alpha,\beta)) = m_s^1 + \alpha/s$ *then*

$$\xi_{j1}(\alpha,\beta) = \max_k\{\max_{l,k=u+v+1}(c_{j1}^k+\alpha, b_{j2}^u+c_{2l}^v+ a_{l1}+\alpha+\beta) - k(m_s^1+\alpha/s)\}.$$

2. Let $\lambda(A(\alpha,\beta)) = m_s^2 + \beta/s$ *then*

$$\xi_{j2}(\alpha,\beta) = \max_k\{\max_{l,k=u+v+1}(b_{j2}^k+\beta, c_{j1}^u+b_{1l}^v+ a_{l2}+\alpha+\beta) - k(m_s^2+\beta/s)\}.$$

3. Let $\lambda(A(\alpha,\beta)) = m_s^{1,2} + (\alpha+\beta)/s$ *then*

$$\xi_{j1}(\alpha,\beta) = \max_k\{\max_{l,k=u+v+1}(c_{j1}^k+\alpha, b_{j2}^u+c_{2l}^v+ a_{l1}+\alpha+\beta) - k(m_s^{1,2}+(\alpha+\beta)/s)\}$$

or

$$\xi_{j2}(\alpha,\beta) = \max_k\{\max_{l,k=u+v+1}(b_{j2}^k+\beta, c_{j1}^u+b_{1l}^v+ a_{l2}+\alpha+\beta) - k(m_s^{1,2}+(\alpha+\beta)/s)\}.$$

Proof. Let us assume that $\lambda(A(\alpha,\beta)) = m_s^1+\alpha/s$. We denote by P the matrix $A(\alpha,\beta)$ and w_{j1} the weight of heaviest path $p = (j,j_1,\dots,j_r,1)$ from j to 1 in D_P. Suppose now $w_{j1} > \xi_{j1}(\alpha,\beta) = \max_k\{\max_{l,k=u+v+1}(c_{j1}^k+\alpha, b_{j2}^u+c_{2l}^v+a_{l1}+\alpha+\beta) - k(m_s^1+\alpha/s)\}$. The last inequality is equivalent to the formula $w_{j1} > \xi_{j1}(\alpha,\beta) = \max_{l,k=u+v+1}\{(c_{j1}^k+\alpha, b_{j2}^u+c_{2l}^v+a_{l1}+\alpha+\beta) - k(m_s^1+\alpha/s)\}$ for all $k\in N$. Then for every k we have two possibilities:

(a) $2 \in p$, (b) $2 \notin p$.
In the case (a),

$$w_{j1} = p_{jj_1} + \cdots + p_{j_r 1} =$$
$$a_{jj_1} + \cdots + a_{j_v 2} + a_{2 j_{v+1}} + \cdots + a_{j_{r-1} j_r} + a_{j_r 1} + \alpha + \beta -$$
$$(r+1)(m_s^1 + \alpha/s) \leq$$
$$b_{j2}^u + c_{2 j_r}^t + a_{j_r 1} + \alpha + \beta - (r+1)(m_s^1 + \alpha/s) \leq \xi_{j1}(\alpha, \beta).$$

In the case (b),

$$w_{j1} = p_{jj_1} + \cdots + p_{j_r 1} =$$
$$a_{jj_1} + \cdots + a_{j_{r-1} j_r} + a_{j_r 1} + \alpha - (r+1)(m_s^1 + \alpha/s) \leq$$
$$c_{j1}^t + \alpha - (r+1)(m_s^1 + \alpha/s) \leq \xi_{j1}(\alpha, \beta).$$

The case (a) and (b) lead to a contradiction. Analogously as above, it can be proved statements 2. and 3.
□

4.1. Computing procedure

Last sections describe the procedure which computes all eigenvalues and corresponding eigenvectors dependent on parameters α, β. To give the computational complexity of the considered procedure we will use the $O(n^3)$ Karp's, Floyd-Warshall's algorithms and procedure presented in last sections.

Procedure **Biparameter**
Input: A given matrix A
Output: $\lambda(A(\alpha, \beta))$, $\xi_{ij}(\alpha, \beta)$, $i \in N, j \in \{1, 2\}$
1. *Compute* $m_v^j(\alpha, \beta)$ for $j \in \{\{1\}, \{2\}, \{1, 2\}\}$, $v \in N$
2. *Determine* $P_>^j(v)$ for $j \in \{\{1\}, \{2\}, \{1, 2\}\}$
3. *Describe* $\xi_{ij}(\alpha, \beta)$, $i \in N, j \in \{1, 2\}$.

Theorem 4.3. *Procedure* **Biparameter** *works correct and terminates after $O(n^5)$ steps.*

Proof. First step uses the *Theorem 4.1* and works at $O(n^3)$ time. Second step needs $O(n^2)$ times to solve system of linear inequalities in $O(n^3)$ time. Third step uses the values known from first and second steps and works at $O(n^3)$ time. Third step works according to *Theorem 4.2* in $O(n^3)$ time. Then this procedure has the best worst-case performance $O(n^5)$.
□

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BIOGRAPHIES

Martin Gavalec graduated in mathematics from Comenius University. At present he is a professor at the Faculty of Informatics and Management, University of Hradec Králové. His scientific research is focusing on combinatorial optimization and management.

Ján Plavka graduated in discrete mathematics from P. J. Šafarik University. At present he is an Associate professor at the Faculty of Electrical Engineering and Informatics, Technical University of Košice . His scientific research is focusing on computer science, algorithms and complexity. In addition, he also investigates questions related with the discrete dynamic systems.