

## THE MATHEMATICAL MODEL AND SELECTED TRANSIENT STATES OF THREE-PHASE INDUCTION MACHINE WITH SATURATED MAGNETIC CIRCUIT

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### ABSTRACT

*A development of mathematical models of control systems is a fundamental question during realization of analysis and synthesis tasks. The development is significant nowadays as majority of tasks is solved by mathematical modelling and digital simulations [1 – 8]. In electric power units the motor supplied via inverter forms controlled system. The circuit models are commonly used in mathematical description of electric drives despite the fact that there is a quick development of both field models describing electric machines and simulation methods considering these models. The circuit models include motor, inverter and control system.*

*The mathematical model together with mathematical and numerical studies of induction motor is presented in the paper. The model considers saturation of motor magnetic circuit, the skin effect occurring in bars of squirrel-cage rotor and moment of friction in bearings of motor as a function of rotational speed of rotor. This model reproduces phenomena occurring in three-phase induction machine and may be applied in order to analyze the transient states. It also may be applied for synthesis of appropriate control systems.*

*Digital simulations of starting the induction motor were made using presented mathematical model. Transient responses and trajectories obtained as a result of simulations are given in the paper. The impact of magnetic circuit saturation on induction motor parameters is also discussed.*

**Keywords:** machine, inductance, magnetic, saturation, modelling, simulation, analysis, synthesis, control.

### 1. INTRODUCTION

A development of mathematical models of control systems is a fundamental question during realization of analysis and synthesis tasks. The development is significant nowadays as majority of tasks is solved by mathematical modelling and digital simulations [1 – 8]. In electric power units the motor supplied via inverter forms controlled system (plant). The circuit models are commonly used in mathematical description of electric drives despite the fact that there is a quick development of both field models describing electric machines and simulation methods considering these models. The circuit models include motor, inverter and control system.

Construction of circuit models of electric machines is possible if some assumptions are considered. The assumptions concerning three-phase induction machine are as follows [9, 10, 11]:

- spatially distributed stator windings and rotor windings (or squirrel-cage of rotor) are replaced by concentrated windings,
- the symmetrical three-phase machine is considered,
- the uniformity of air-gap is assumed,
- an influence of anisotropy, magnetic hysteresis and eddy currents is omitted,
- the higher harmonics of spatially distributed magnetic field in air-gap are omitted and only fundamental harmonic is taken into consideration,
- resistances and leakage reactances of windings are assumed to be constant parameters.

The circuit models underlie mathematical modelling the induction machine described with differential equations.

### 2. MATHEMATICAL MODEL OF INDUCTION MACHINE WITH VARIABLE MUTUAL INDUCTANCE

The induction machine contains six magnetically coupled circuits according to considered assumptions. There are three circuits of stator immovable in space and three circuits of rotor. Matrix equations of transient electromagnetic processes in magnetically coupled circuits of symmetrical three-phase induction machine may be expressed in phase coordinates as follows:

$$U_{ABC} = R_s I_{ABC} + \frac{d\Psi_{ABC}}{dt} \quad (1)$$

$$U_{abc} = R_r I_{abc} + \frac{d\Psi_{abc}}{dt} \quad (2)$$

where  $U_{ABC} = [u_A \ u_B \ u_C]^T$  is a vector including phase voltages at a point of connection of induction machine to the power grid,  $I_{ABC} = [i_A \ i_B \ i_C]^T$  is vector including phase currents of stator,  $U_{abc} = [u_a \ u_b \ u_c]^T$  is vector including phase voltages across the rotor windings,  $I_{abc} = [i_a \ i_b \ i_c]^T$  is vector including phase currents in rotor circuits,  $\Psi_{ABC} = [\psi_A \ \psi_B \ \psi_C]^T$ ,  $\Psi_{abc} = [\psi_a \ \psi_b \ \psi_c]^T$  are vectors including phase fluxes of stator and rotor,  $R_s$ ,  $R_r$  – are resistances of phase winding of stator and rotor,  $T$  is matrix transposition. The fluxes are functions of currents:

$$\begin{bmatrix} \Psi_{ABC} \\ \Psi_{abc} \end{bmatrix} = \begin{bmatrix} L_s & L_{sr} \\ L_{rs} & L_r \end{bmatrix} \begin{bmatrix} I_{ABC} \\ I_{abc} \end{bmatrix},$$

where

$$L_s = \begin{bmatrix} L_{\sigma s} + M_A & -\frac{1}{2}M_B & -\frac{1}{2}M_C \\ -\frac{1}{2}M_A & L_{\sigma s} + M_B & -\frac{1}{2}M_C \\ -\frac{1}{2}M_A & -\frac{1}{2}M_B & L_{\sigma s} + M_C \end{bmatrix}$$

$$L_r = \begin{bmatrix} L_{\sigma r} + M_a & -\frac{1}{2}M_b & -\frac{1}{2}M_c \\ -\frac{1}{2}M_a & L_{\sigma r} + M_b & -\frac{1}{2}M_c \\ -\frac{1}{2}M_a & -\frac{1}{2}M_b & L_{\sigma r} + M_c \end{bmatrix}$$

$$L_{sr} = \begin{bmatrix} M_a \cos \gamma'_m & M_b \cos \left( \gamma'_m + \frac{2\pi}{3} \right) \\ M_a \cos \left( \gamma'_m - \frac{2\pi}{3} \right) & M_b \cos \gamma'_m \\ M_a \cos \left( \gamma'_m + \frac{2\pi}{3} \right) & M_b \cos \left( \gamma'_m - \frac{2\pi}{3} \right) \end{bmatrix}$$

$$\begin{bmatrix} M_c \cos \left( \gamma'_m - \frac{2\pi}{3} \right) \\ M_c \cos \left( \gamma'_m + \frac{2\pi}{3} \right) \\ M_c \cos \gamma'_m \end{bmatrix}$$

$$L_{rs} = \begin{bmatrix} M_A \cos \gamma'_m & M_B \cos \left( \gamma'_m - \frac{2\pi}{3} \right) \\ M_A \cos \left( \gamma'_m + \frac{2\pi}{3} \right) & M_B \cos \gamma'_m \\ M_A \cos \left( \gamma'_m - \frac{2\pi}{3} \right) & M_B \cos \left( \gamma'_m + \frac{2\pi}{3} \right) \end{bmatrix}$$

$$\begin{bmatrix} M_C \cos \left( \gamma'_m + \frac{2\pi}{3} \right) \\ M_C \cos \left( \gamma'_m - \frac{2\pi}{3} \right) \\ M_C \cos \gamma'_m \end{bmatrix}$$

$\gamma'_m = p_b \gamma_m$ ,  $p_b$  is number of couple pairs,  $\gamma_m$  is angle of rotor rotation,  $\gamma_m/dt = \omega_m$ ,  $\omega_m$  is angular velocity of rotor,  $L_{\sigma s}$ ,  $L_{\sigma r}$  are leakage inductances of stator and rotor phase windings. The mutual inductance  $M_k = M(\psi_{mk})$  for  $k = A, B, C, a, b, c$  is nonlinear function of main flux.

(3) The following dependence may be applied in order to represent the mutual inductance [1, 2, 3]:

$$M(\psi_{mk}) = \frac{M(0)}{\sqrt{b(\psi_{mk}/\psi_{mn})^{2a} + 1}} \tag{4}$$

where

$$\Psi_{mABC} = \Psi_{ABC} - L_{\sigma s} i_{ABC}, \quad \Psi_{mabc} = \Psi_{abc} - L_{\sigma r} i_{abc},$$

$$b = \frac{M^2(0)}{M^2(\psi_{mn})} - 1, \quad M(0) = \frac{2}{3} L_m(0) \approx \frac{U}{3\pi f_n I_0} - L_{\sigma s},$$

provided that  $U \ll U_n$ , while value of parameter  $a$  should be determined experimentally.

The equation of rotor motion is given below:

$$J \frac{d\omega_m}{dt} = T_e - T_m \tag{5}$$

where  $J$  is moment of inertia of rotor and connected to its rotating elements,  $T_m$  is torque of external forces (load torque, moment of friction) applied to the shaft. The electromagnetic torque  $T_e$  exerting an influence on rotor (output torque) is defined as following partial derivative:

$$T_e = p_b \frac{\partial W_{em}}{\partial \gamma'_m} \tag{6}$$

where  $W_{em}$  is energy of magnetic fields which is expressed as follows:

$$W_{em} = \frac{1}{2} \sum_k i_k \Psi_k, \quad k = A, B, C, a, b, c \tag{7}$$

The electromagnetic torque may be expressed as equation given below due to the self-inductances and mutual inductances included in matrix  $L_s$  and matrix  $L_r$  are independent of position angle between stator winding and rotor winding:

$$T_e = \frac{p_b}{2} \left( I_{ABC}^T \frac{\partial}{\partial \gamma'_m} L_{sr} I_{abc} + I_{abc}^T \frac{\partial}{\partial \gamma'_m} L_{rs} I_{ABC} \right) \tag{8}$$

The system of differential equations containing periodical coefficients depending on angle  $\gamma'_m$  occurs if expressions included in the matrix (3) substitute the flux vectors in voltage equations (1), (2) describing induction machine. The Park's transformation (9) [9, 10] is used in order to eliminate periodical coefficients from equations describing induction machine. The variables of two-phase induction machine are obtained as a result of Park's transformation instead of variables of real three-phase machine.

$$\mathbf{P}(\gamma'_m) = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos \gamma'_m & \cos\left(\gamma'_m + \frac{2\pi}{3}\right) & \cos\left(\gamma'_m - \frac{2\pi}{3}\right) \\ \sin \gamma'_m & \sin\left(\gamma'_m + \frac{2\pi}{3}\right) & \sin\left(\gamma'_m - \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad (9)$$

Above matrix transforms three-axis coordinate system representing phase windings of induction machine into two-axis Cartesian coordinate system.

There is no zero-sequence current in three-phase systems without neutral conductor, i. e.:

$$i_A + i_B + i_C = i_a + i_b + i_c = 0 \quad (10)$$

From dependences (3) and (10) it follows that:

$$\psi_A + \psi_B + \psi_C = L_{\sigma s}(i_A + i_B + i_C) = 0$$

$$\psi_a + \psi_b + \psi_c = L_{\sigma r}(i_a + i_b + i_c) = 0$$

The row vector in Park's matrix (9) referring to zero-sequence component may be omitted in that case.

The new auxiliary variables applied instead of instantaneous values of stator and rotor phase currents simplify analysis of induction machine described by equations containing nonlinear dependences between inductances and fluxes:

$$x_k = M_k i_k, \quad k = A, B, C, a, b, c \quad (11)$$

Thus, the matrix dependence (3) considering new variables  $x_k$  may be expressed as:

$$\begin{bmatrix} \Psi_{ABC} \\ \Psi_{abc} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_s & \mathbf{A}_{sr} \\ \mathbf{A}_{rs} & \mathbf{A}_r \end{bmatrix} \begin{bmatrix} \mathbf{X}_{ABC} \\ \mathbf{X}_{abc} \end{bmatrix}, \quad (12)$$

where

$$\mathbf{A}_s = \begin{bmatrix} 1 + \sigma_A & -1/2 & -1/2 \\ -1/2 & 1 + \sigma_B & -1/2 \\ -1/2 & -1/2 & 1 + \sigma_C \end{bmatrix}$$

$$\mathbf{A}_r = \begin{bmatrix} 1 + \sigma_a & -1/2 & -1/2 \\ -1/2 & 1 + \sigma_b & -1/2 \\ -1/2 & -1/2 & 1 + \sigma_c \end{bmatrix}$$

$$\sigma_A = L_{\sigma s} / M_A, \quad \sigma_B = L_{\sigma s} / M_B, \quad \sigma_C = L_{\sigma s} / M_C,$$

$$\sigma_a = L_{\sigma r} / M_a, \quad \sigma_b = L_{\sigma r} / M_b, \quad \sigma_c = L_{\sigma r} / M_c,$$

$$\mathbf{A}_{sr} = \mathbf{A}_{sr}^T = \begin{bmatrix} \cos \gamma'_m & \cos\left(\gamma'_m + \frac{2\pi}{3}\right) & \cos\left(\gamma'_m - \frac{2\pi}{3}\right) \\ \cos\left(\gamma'_m - \frac{2\pi}{3}\right) & \cos \gamma'_m & \cos\left(\gamma'_m + \frac{2\pi}{3}\right) \\ \cos\left(\gamma'_m + \frac{2\pi}{3}\right) & \cos\left(\gamma'_m - \frac{2\pi}{3}\right) & \cos \gamma'_m \end{bmatrix}$$

The following matrix equation related to immovable Cartesian coordinate system  $0\alpha\beta$  is obtained as a result of Park's transformation:

$$\begin{bmatrix} \Psi_{s\alpha\beta} \\ \Psi_{r\alpha\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{P}(0)\mathbf{A}_s\mathbf{P}^T(0) & \mathbf{P}(0)\mathbf{A}_{sr}\mathbf{P}^T(\gamma'_m) \\ \mathbf{P}(\gamma'_m)\mathbf{A}_{rs}\mathbf{P}^T(0) & \mathbf{P}(\gamma'_m)\mathbf{A}_r\mathbf{P}^T(\gamma'_m) \end{bmatrix} \begin{bmatrix} \mathbf{X}_{s\alpha\beta} \\ \mathbf{X}_{r\alpha\beta} \end{bmatrix} \quad (13)$$

where

$$\mathbf{P}(0)\mathbf{A}_s\mathbf{P}^T(0) \approx \begin{bmatrix} \frac{3}{2} + \frac{4\sigma_A + \sigma_B + \sigma_C}{6} & 0 \\ 0 & \frac{3}{2} + \frac{\sigma_B + \sigma_C}{2} \end{bmatrix}$$

$$\mathbf{P}(\gamma'_m)\mathbf{A}_r\mathbf{P}^T(\gamma'_m) \approx \left(\frac{3}{2} + \frac{\sigma_a + \sigma_b + \sigma_c}{3}\right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{P}(0)\mathbf{A}_{sr}\mathbf{P}^T(\gamma'_m) = \mathbf{P}(\gamma'_m)\mathbf{A}_{rs}\mathbf{P}^T(0) \approx \frac{3}{2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Psi_{s\alpha\beta} = [\psi_{s\alpha} \quad \psi_{s\beta}]^T, \quad \Psi_{r\alpha\beta} = [\psi_{r\alpha} \quad \psi_{r\beta}]^T,$$

$$\mathbf{X}_{s\alpha\beta} = [x_{s\alpha} \quad x_{s\beta}]^T, \quad \mathbf{X}_{r\alpha\beta} = [x_{r\alpha} \quad x_{r\beta}]^T.$$

The equations describing transient electromagnetic processes in induction machine without neutral conductor and without additional elements in rotor circuits (including squirrel-cage motor) are as follows:

$$\frac{d\Psi_{s\alpha\beta}}{dt} = -R_s \mathbf{I}_{s\alpha\beta} + \mathbf{U}_{s\alpha\beta} \quad (14)$$

$$\frac{d\Psi_{r\alpha\beta}}{dt} = -R_r \mathbf{I}_{r\alpha\beta} + p_b \omega_m \mathbf{J} \Psi_{r\alpha\beta} \quad (15)$$

$$\text{where } \mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The components of flux vectors contained in the equations (15), (16) are calculated with application of chosen formula of numerical integration. Subsequently the components of vectors  $\mathbf{X}_{s\alpha\beta}$ ,  $\mathbf{X}_{r\alpha\beta}$  derived from (13) are

calculated and afterwards mentioned components are transformed to three-phase system to calculate the phase currents:  $i_k = x_k / M_k$ ,  $k = A, B, C, a, b, c$ . The values of both phase currents and phase fluxes allow calculating main fluxes and mutual inductances for each phase winding.

Output torque may be determined if following dependences will be taken into formula (8):

$$\begin{aligned} \mathbf{I}_{ABC}^T &= \mathbf{I}_{s\alpha\beta}^T \mathbf{P}(0), \quad \mathbf{L}_{sr} \mathbf{I}_{abc} = \mathbf{A}_{sr} \mathbf{P}^T(\gamma'_m) \mathbf{X}_{r\alpha\beta}, \\ \mathbf{I}_{abc}^T &= \mathbf{I}_{r\alpha\beta}^T \mathbf{P}(\gamma'_m), \quad \mathbf{L}_{rs} \mathbf{I}_{ABC} = \mathbf{A}_{rs} \mathbf{P}^T(0) \mathbf{X}_{s\alpha\beta}, \end{aligned}$$

then:

$$T_e = \frac{3pb}{4} (i_{r\alpha} x_{s\beta} - i_{r\beta} x_{s\alpha} + i_{s\beta} x_{r\alpha} - i_{s\alpha} x_{r\beta}) \quad (16)$$

Presented mathematical model taking into consideration the saturation of magnetic circuits reproduces phenomena occurring in three-phase induction machine. The model may be applied in order to analyze the transient states. It also may be applied for synthesis of appropriate control systems.

### 3. EXAMPLES OF TRANSIENT RESPONSES AND TRAJECTORIES

The digital simulations of starting an induction motor of 55kW were made with application of above presented mathematical model. The following transient responses and trajectories were obtained as a result of the simulations. The impact of magnetic circuit saturation on induction motor parameters was taken into consideration (Fig. 1 – 8). The digital simulation of starting the motor with application of mathematical model omitting the impact of magnetic circuit saturation was also made for comparison (Fig. 9 – 15).

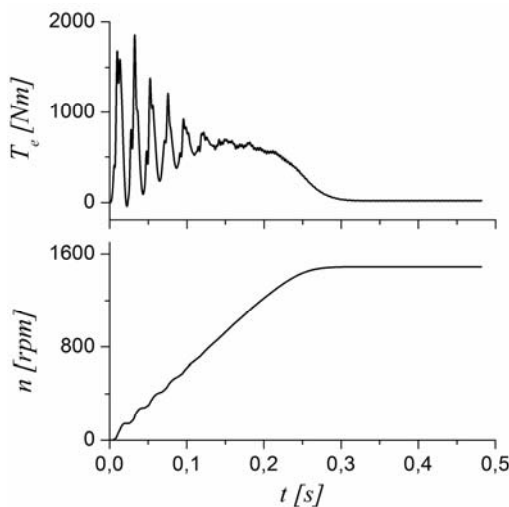


Fig. 1 Transient responses of motor including output torque  $T_e$  and rotational speed  $n$  of rotor

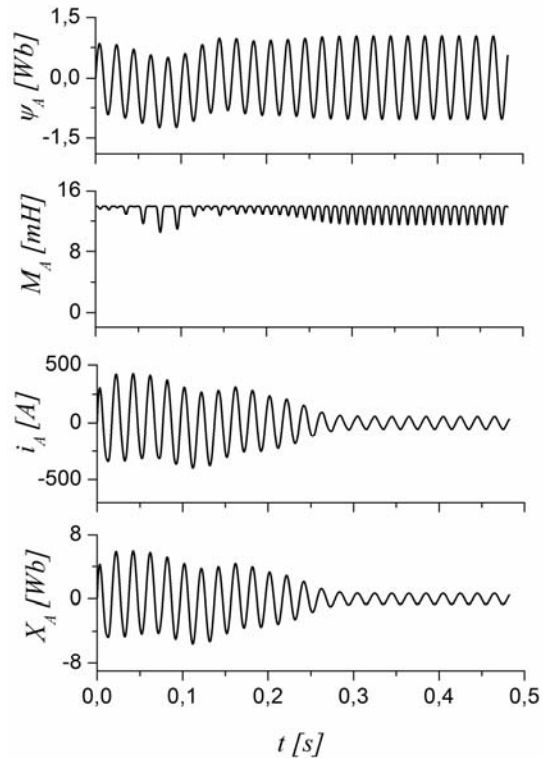


Fig. 2 Transient responses of motor including phase variables of stator (flux  $\psi_A$ , mutual inductance  $M_A$ , current  $i_A$  and variable  $X_A$ )

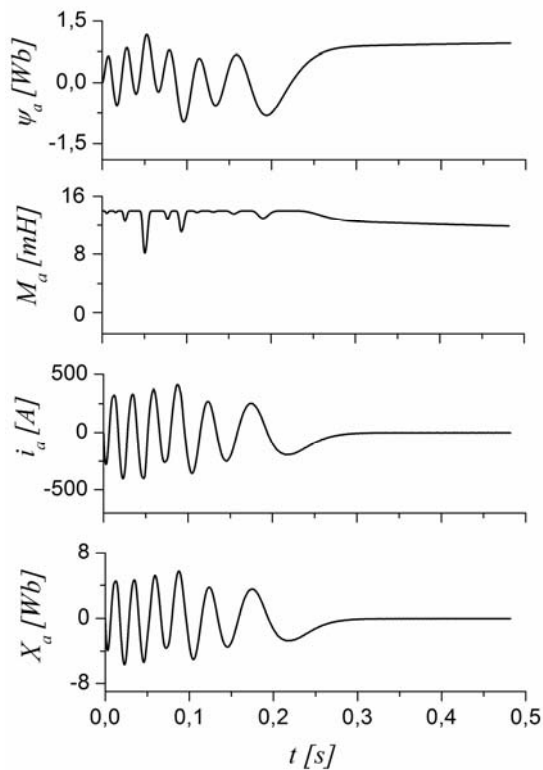
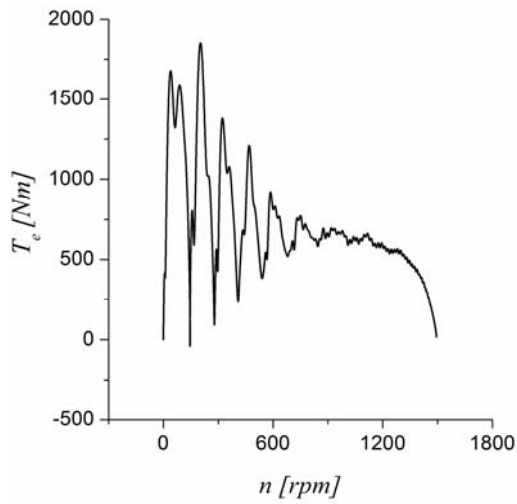
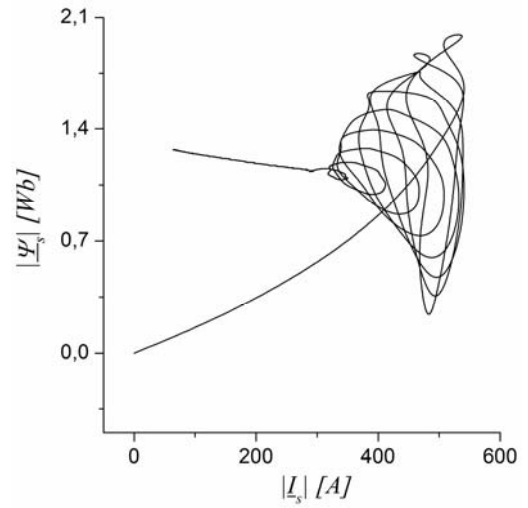


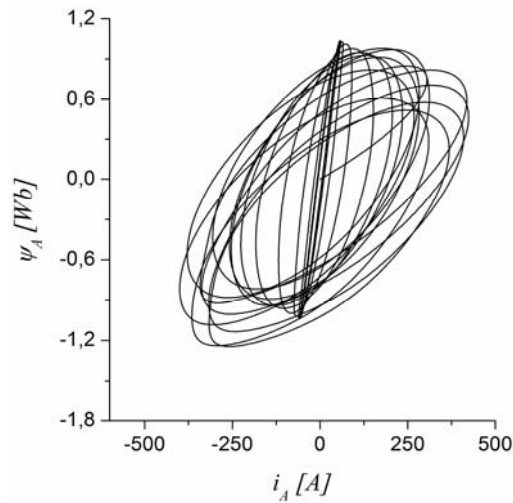
Fig. 3 Transient responses of motor including phase variables of rotor (flux  $\psi_a$ , mutual inductance  $M_a$ , current  $i_a$  and variable  $X_a$ )



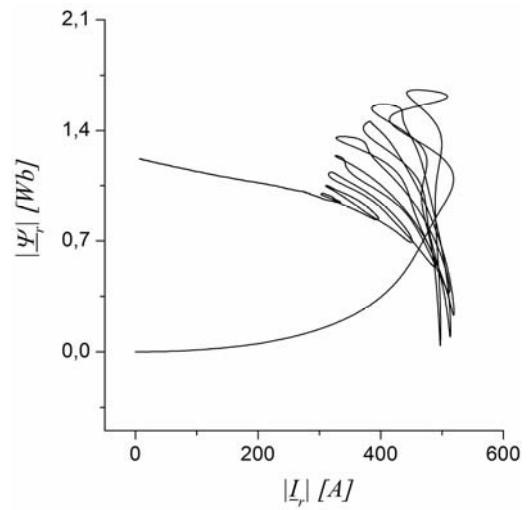
**Fig. 4** Trajectory of output torque of motor as a function of rotational speed of rotor



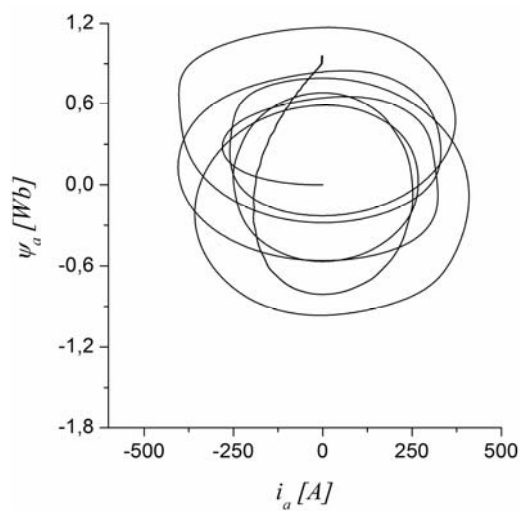
**Fig. 7** Trajectory of absolute values of stator flux vector as a function of stator current vector



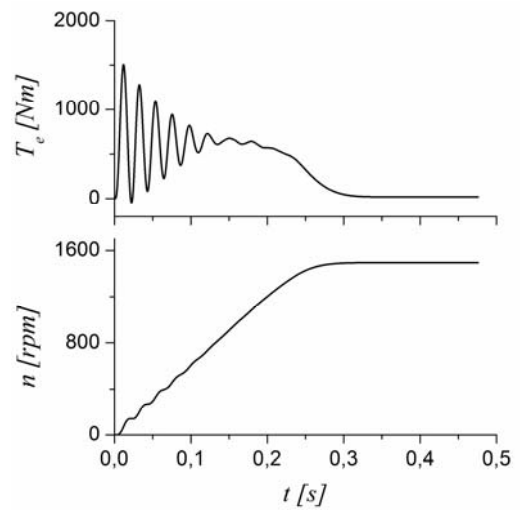
**Fig. 5** Trajectory of phase stator flux  $\psi_A$  as a function of phase stator current  $i_A$



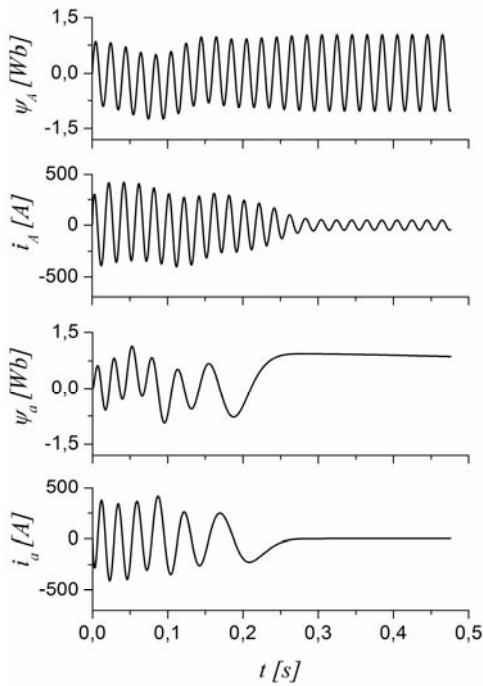
**Fig. 8** Trajectory of absolute values of rotor flux vector as a function of rotor current vector



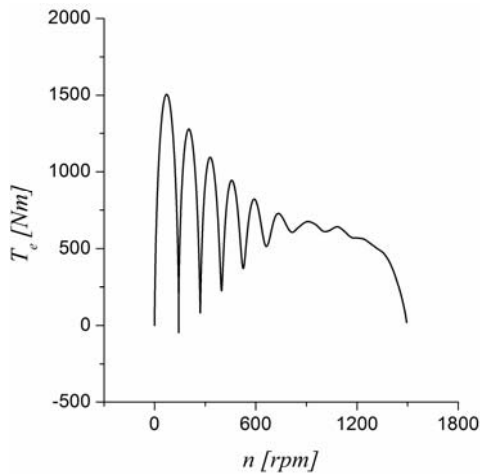
**Fig. 6** Trajectory of phase rotor flux  $\psi_a$  as a function of phase rotor current  $i_a$



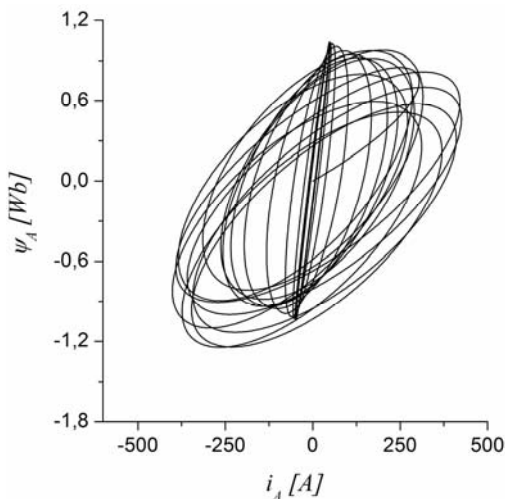
**Fig. 9** Transient responses of motor including output torque  $T_e$  and rotational speed  $n$  of rotor



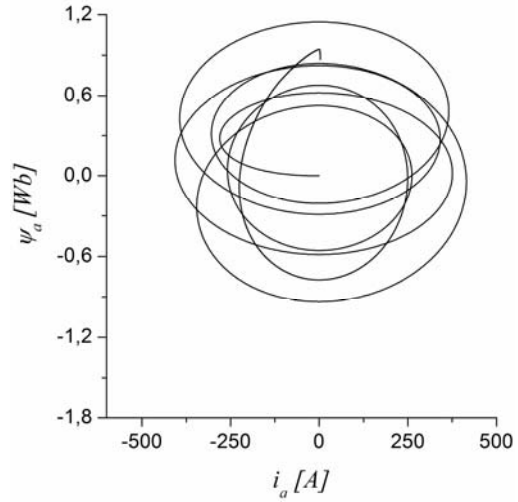
**Fig. 10** Transient responses of motor including phase variables of stator (flux  $\psi_A$  and current  $i_A$ ) and phase variables of rotor (flux  $\psi_a$  and current  $i_a$ )



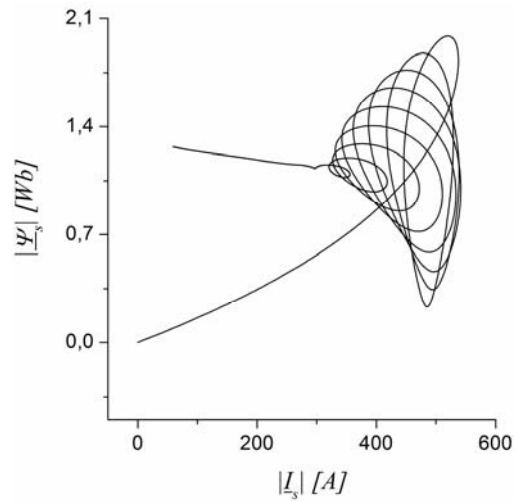
**Fig. 11** Trajectory of output torque of motor as a function of rotational speed of rotor



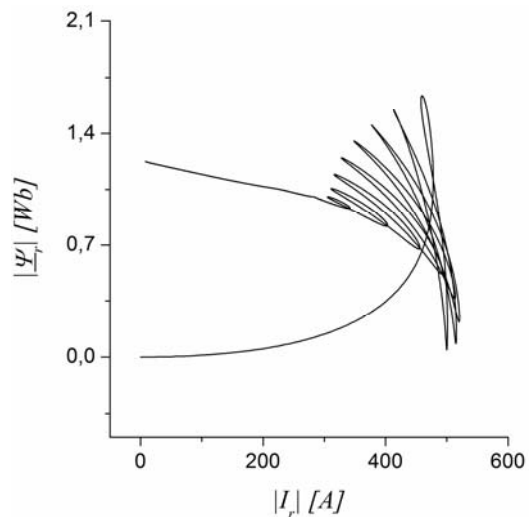
**Fig. 12** Trajectory of phase stator flux  $\psi_A$  as a function of phase stator current  $i_A$



**Fig. 13** Trajectory of phase rotor flux  $\psi_a$  as a function of phase rotor current  $i_a$



**Fig. 14** Trajectory of absolute values of stator flux vector as a function of stator current vector



**Fig. 15** Trajectory of absolute values of rotor flux vector as a function of rotor current vector

#### 4. CONCLUSION

The mathematical model together with mathematical and numerical studies of induction motor is presented in the paper. The mathematical model considers some real phenomena e. g. saturation of motor magnetic circuit, the skin effect occurring in bars of squirrel-cage rotor and moment of friction in bearings of motor as a function of rotational speed of rotor.

The saturation of main magnetic circuit of motor causes additional distortions of both transient phase currents and transient output torque (Fig. 1, 4 to 8). The higher harmonics occur in mentioned variables as a result of taking into consideration a nonlinear magnetization curve. The saturation of main magnetic circuit increases extreme values of output torque (compare Fig. 4 and Fig. 11). The significant increase of magnetizing current takes place when nominal value of supply voltage is exceeded. That is additional disadvantage of magnetic circuit saturation.

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